Bubble–type solutions of nonlinear singular problem*

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Abstract

The paper describes the set of all solutions of the singular initial problems

$$(p(t)u')' = p(t)f(u), \quad u(0) = B,\ u'(0) = 0,$$

on the half–line $[0, \infty)$. Here $B < 0$ is a parameter, $p(0) = 0$ and $p'(t) > 0$ on $(0, \infty)$, $f(L) = 0$ for some $L > 0$ and $xf(x) < 0$ if $x < L$, $x \neq 0$. By means of this result, the existence of a strictly increasing solution of this problem satisfying $u(\infty) = L$ is proved under some additional assumptions. In particular cases this homoclinic solution determines an increasing mass density in centrally symmetric gas bubbles which are surrounded by an external liquid with the density L.

Mathematics Subject Classification 2000: 34B16, 34B40.

Key words: Singular ordinary differential equation of the second order, homoclinic solution, time singularities, unbounded domain.

1 Introduction

We investigate a singular boundary value problem which originates from the Cahn-Hillard theory in hydrodynamics. If ρ is the density, $\mu(\rho)$ the chemical potential of a non-homogeneous fluid and the motion of the fluid is zero, the state of the fluid in $\mathbb{R}^N$ is described by the equation

$$\gamma \Delta \rho = \mu(\rho) - \mu_0, \tag{1}$$

*Supported by the Council of Czech Government MSM 6198959214 and by the grant No. A100190703 of the Grant Agency of the Academy of Sciences of the Czech Republic.

where γ and μ_0 are suitable constants. See [4]–[7] and [9]. When searching for a solution with the spherical symmetry which depends only on one variable r, equation (1) is reduced to the following ordinary differential equation

$$\gamma(\rho'' + \frac{N-1}{r}\rho') = \mu(\rho) - \mu_0, \quad r \in (0, \infty). \tag{2}$$

In fact, together with the boundary conditions

$$\rho'(0) = 0, \quad \lim_{r \to \infty} \rho(r) = \rho_\ell > 0, \tag{3}$$

equation (2) describes the formation of microscopical bubbles in a fluid, in particular, vapor inside one liquid. The first condition in (3) follows from the central symmetry and it is necessary for the smoothness of solutions of the singular equation (2) at $r = 0$. The second condition in (3) means that the bubble is surrounded by an external liquid with the density ρ_ℓ.

Let $N = 3$. In the simplest models for non-homogeneous fluids, the chemical potential μ is a third degree polynomial with three distinct real roots. After some substitution (see [9]), problem (2), (3) is reduced to the form

$$(t^2 u')' = 4\lambda^2 t^2 (u+1) u (u - \xi), \tag{4}$$

$$u'(0) = 0, \quad u(\infty) = \xi, \tag{5}$$

where $\lambda \in (0, \infty)$ and $\xi \in (0, 1)$ are parameters. If there exists an increasing solution of problem (4), (5) (having just one zero), many important physical properties of the bubbles depend on them. In particular, the gas density inside the bubble, the bubble radius and the surface tension. Numerical investigation of the problem can be found in [4], [7]–[9]. Note that the same boundary value problems arise in the nonlinear field theory [5].

2 Formulation of problem

We investigate possible generalizations of problem (4), (5). In particular, we consider a singular boundary value problem on the half–line of the form

$$(p(t)u')' = p(t) f(u), \tag{6}$$

$$u'(0) = 0, \quad u(\infty) = L, \tag{7}$$

where $L \in (0, \infty)$. We prove the existence of a strictly increasing solution of problem (6), (7) having just one zero and belonging to $C^1([0, \infty)) \cap C^2((0, \infty))$. Problem (6), (7) can be transformed onto a problem about the existence of a positive solution on the half–line. For $p(t) = t^k$, $k \in \mathbb{N}$ and for $p(t) = t^k$, $k \in (1, \infty)$, such problem was solved by variational methods in [1] and [2], respectively. Related problems were solved e.g. in [3] and [10]. Here we deal with a more general function p and we omit some assumptions for f.

In what follows we assume

$$f \text{ is locally Lipschitz on } (-\infty, L],\ f(L) = 0, \tag{8}$$

$$xf(x) < 0 \text{ for } x < L,\ x \neq 0, \tag{9}$$

$$\text{there exists } \bar{B} < 0 \text{ such that } F(\bar{B}) = F(L), \text{ where } F(x) = -\int_0^x f(z)\,\mathrm{d}z, \tag{10}$$

$$p \in C^1((0,\infty)) \cap C([0,\infty)),\ p(0) = 0, \tag{11}$$

$$p'(t) > 0,\ t \in (0,\infty), \quad \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0. \tag{12}$$

For example functions

$$\begin{aligned} &p(t) = t^k,\ p(t) = t^k \ln(t+1),\ k > 0,\\ &p(t) = t + \alpha \sin t,\ \alpha \in (-1,1),\\ &p(t) = \frac{t^k}{1+t^l},\ k \geq l > 0, \end{aligned}$$

satisfy (11) and (12).

Remark 1 According to (8) and (9), we have $f(0) = 0$. So, f has just two zeros in $(-\infty, L]$. The case when f has more that two zeros was solved in [11], [12]. Assumptions (9), (10) yield that F is continuous on $(-\infty, L]$, decreasing on $(-\infty, 0)$, increasing on $(0, L)$ and

$$F(B) > F(L) \text{ for } B \in (-\infty, \bar{B}), \quad F(B) < F(L) \text{ for } B \in (\bar{B}, 0]. \tag{13}$$

3 Lemmas

Define auxiliary functions

$$\tilde{f}(x) = \begin{cases} 0 & \text{for} \quad x > L, \\ f(x) & \text{for} \quad x \leq L, \end{cases} \qquad \tilde{F}(x) = -\int_0^x \tilde{f}(z)\,\mathrm{d}z,\ x \in \mathbb{R}. \tag{14}$$

Choose $B < 0$ and consider an initial problem

$$(p(t)u')' = p(t)\tilde{f}(u), \tag{15}$$

$$u(0) = B, \quad u'(0) = 0. \tag{16}$$

Definition 2 Let $c \in (0,\infty]$. A function $u \in C^1([0,c)) \cap C^2((0,c))$ satisfying equation (15) on $(0,c)$ and fulfilling conditions (16) is called a solution of problem (15), (16) on $[0,c)$.

Lemma 3 *Let $B < 0$. Problem* (15), (16) *has a unique solution u on $[0, \infty)$, such that*

$$u(t) \geq B \text{ for } t \in [0, \infty). \tag{17}$$

Further, for each $b > 0$, $B_0 < 0$ and each $\delta \in (0, |B_0|/2)$, there exists $M = M(b, B_0, \delta) > 0$ such that

$$|u(t)| + |u'(t)| \leq M, \ t \in [0, b], \quad \int_0^b \frac{p'(s)}{p(s)} |u'(s)| \, \mathrm{d}s \leq M \tag{18}$$

holds for each solution u of problem (15), (16) *with $B \in (B_0 - \delta, B_0 + \delta)$.*

Proof. Step 1. (A priori estimate of solutions) Let u be a solution of problem (15), (16) on $[0, c) \subset [0, \infty)$. By (15), we have

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) - \tilde{f}(u(t)) = 0 \quad \text{for } t \in (0, c),$$

and multiplying by u' and integrating between 0 and t, we get

$$\frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u'^2(s) \, \mathrm{d}s + \tilde{F}(u(t)) = \tilde{F}(B), \ t \in (0, c). \tag{19}$$

Let $u(t_1) < B$ for some $t_1 \in (0, c)$. Then (19) yields $\tilde{F}(u(t_1)) \leq \tilde{F}(B)$, which is not possible, because $\tilde{F}$ is decreasing on $(-\infty, 0)$. Therefore $u(t) \geq B$ for $t \in [0, c)$.
Step 2. (Local solution u of an auxiliary problem) By (8), we can find the Lipschitz constant $K > 0$ for f on $[B, L]$. Put $\varphi(t) = \frac{1}{p(t)} \int_0^t p(s) \, \mathrm{d}s$, $t > 0$. Having in mind (12), we see that

$$0 < \varphi(t) \leq t \quad \text{for } t \in (0, \infty), \ \lim_{t \to 0^+} \varphi(t) = 0. \tag{20}$$

Choose $\eta > 0$ such that

$$\int_0^\eta \varphi(t) \, \mathrm{d}t < \frac{1}{2K},$$

and put

$$\tilde{f}_B(x) = \begin{cases} \tilde{f}(x) & \text{for} \quad x \geq B, \\ \tilde{f}(B) & \text{for} \quad x < B. \end{cases}$$

Consider the Banach space $C([0, \eta])$ (with the maximum norm) and define an operator $\mathcal{F} : C([0, \eta]) \to C([0, \eta])$ by

$$(\mathcal{F}u)(t) = B + \int_0^t \frac{1}{p(s)} \int_0^s p(\tau) \tilde{f}_B(u(\tau)) \, \mathrm{d}\tau \, \mathrm{d}s.$$

We have for $u_1, u_2 \in C([0, \eta])$

$$\|\mathcal{F}u_1 - \mathcal{F}u_2\|_{C([0,\eta])} \leq K \|u_1 - u_2\|_{C([0,\eta])} \int_0^\eta \varphi(s) \, \mathrm{d}s < \frac{1}{2} \|u_1 - u_2\|_{C([0,\eta])}.$$

Hence $\mathcal{F}$ is a contraction and the Banach fixed point theorem yields a unique fixed point $u \in C([0,\eta])$ of $\mathcal{F}$. Therefore

$$u'(t) = \frac{1}{p(t)} \int_0^t p(s)\tilde{f}_B(u(s))\,\mathrm{d}s,\ t \in (0,\eta],\ u(0) = B. \tag{21}$$

Let $B_0 < 0$ and $\delta \in (0, |B_0|/2)$. Choose an arbitrary $B \in (B_0 - \delta, B_0 + \delta)$. Due to the definition of $\tilde{f}_B$ there exists $\tilde{M} = \tilde{M}(B_0,\delta) \in (0,\infty)$ such that

$$|\tilde{f}_B(x)| \le \tilde{M} \quad \text{for } x \in \mathbb{R}.$$

Finally, (20) and (21) yield

$$\lim_{t\to 0^+} |u'(t)| \le \tilde{M} \cdot \lim_{t\to 0^+} \varphi(t) = 0.$$

Step 3. (Global solution u of the auxiliary problem) In Step 2 we have proved that there exists a unique solution u of problem (22), (16) on $[0,\eta]$, where

$$(p(t)u')' = p(t)\tilde{f}_B(u). \tag{22}$$

By (20) and (21),

$$|u'(t)| \le \tilde{M}b, \quad |u(t)| \le |B_0| + \delta + \tilde{M}b^2,\ t \in [0,b], \tag{23}$$

for arbitrary $b > 0$. Having in mind that $\tilde{f}_B \in Lip_{loc}(\mathbb{R})$, u can be (uniquely) extended as a solution of equation (22) to each interval, where u is bounded. Since b is arbitrary, u can be extended as a solution of (22) on $[0,\infty)$. Put

$$\psi(t) = \int_t^b \frac{p'(s)}{p^2(s)} \int_0^s p(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,\ t \in (0,b].$$

Then, using the "per partes" integration and (20), we get

$$\psi(t) = \varphi(t) - \varphi(b) + b - t,\ t \in (0,b],$$

$$\lim_{t\to 0^+} \psi(t) = b - \varphi(b) =: \psi_b \in [0,\infty).$$

Integrating (22) over $(0,t)$ and multiplying by $1/p(t)$ we derive

$$|u'(t)| \le \frac{1}{p(t)} \int_0^t p(s)|\tilde{f}_B(u(s))|\,\mathrm{d}s, \quad t \in (0,b].$$

Multiplying by $p'(t)/p(t)$ and integrating it over $(0,b)$, we get

$$\int_0^b \frac{p'(t)}{p(t)}|u'(t)|\,\mathrm{d}t \le \int_0^b \frac{p'(t)}{p^2(t)} \int_0^t p(s)|\tilde{f}_B(u(s))|\,\mathrm{d}s\,\mathrm{d}t \le \tilde{M}\psi_b. \tag{24}$$

Estimates (18) follow from (23) and (24) for

$$M = \max\{\tilde{M}b + |B_0| + \delta + \tilde{M}b^2, \tilde{M}\psi_b\}.$$

Step 4. (Global solution u of problem (15), (16)) If u is a solution of problem (22), (16) on $[0,\infty)$, then we can prove as in Step 1, that $u(t) \geq B$ on $[0,\infty)$, and hence u fulfils equation (15) on $[0,\infty)$. If u is a solution of problem (15), (16) on $[0,c) \subset [0,\infty)$, then, by Step 1, $u(t) \geq B$ on $[0,c)$, and therefore u can be uniquely extended as a solution of equation (22) on $[0,\infty)$. So, solutions of problem (15), (16) and (22), (16) are the same for all $B < 0$. Therefore, due to Step 3, for each $B < 0$ problem (15), (16) has a unique solution on $[0,\infty)$ satisfying (17) and (18).

□

In what follows by a solution of problem (15), (16) we mean a solution on $[0,\infty)$.

Remark 4 Choose $a \geq 0$ and $C \leq L$, and consider the initial conditions

$$u(a) = C, \;\; u'(a) = 0. \tag{25}$$

We can prove as in the proof of Lemma 3 that problem (15), (25) has a unique solution on $[a,\infty)$. In particular, for $C = 0$ and $C = L$, the unique solution of problem (15), (25) is $u \equiv 0$ and $u \equiv L$, respectively.

Lemma 5 *For each $B_0 < 0$, $b > 0$ and each $\epsilon > 0$, there exists $\delta > 0$ such that for any B_1, $B_2 \in [B_0, 0)$*

$$|B_1 - B_2| < \delta \quad \Longrightarrow \quad |u_1(t) - u_2(t)| + |u_1'(t) - u_2'(t)| < \epsilon, \;\; t \in [0,b]. \tag{26}$$

Here u_i is the unique solution of problem (15), (16) *with $B = B_i$, $i = 1,2$.*

Proof. Choose $B_0 < 0$, $b > 0$, $\epsilon > 0$. Let $K > 0$ be the Lipschitz constant for f on $[B_0, L]$. By (15) for $u = u_i$, $i = 1,2$, B_1, $B_2 \in [B_0, 0)$ and by the Fubini theorem

$$\begin{aligned}
|u_1(t) - u_2(t)| &\leq |B_1 - B_2| + \int_0^t \frac{1}{p(s)} \int_0^s p(\tau) |\tilde{f}(u_1(\tau)) - \tilde{f}(u_2(\tau))| \, \mathrm{d}\tau \, \mathrm{d}s \\
&\leq |B_1 - B_2| + \int_0^t \frac{p(s)}{p(s)} \int_0^s |\tilde{f}(u_1(\tau)) - \tilde{f}(u_2(\tau))| \, \mathrm{d}\tau \, \mathrm{d}s \\
&\leq |B_1 - B_2| + \int_0^t \int_0^t |\tilde{f}(u_1(\tau)) - \tilde{f}(u_2(\tau))| \, \mathrm{d}\tau \, \mathrm{d}s \\
&\leq |B_1 - B_2| + Kt \int_0^t |u_1(\tau) - u_2(\tau)| \, \mathrm{d}\tau \\
&\leq |B_1 - B_2| + Kb \int_0^t |u_1(\tau) - u_2(\tau)| \, \mathrm{d}\tau, \;\; t \in [0,b].
\end{aligned}$$

From the Gronwall inequality, we get

$$|u_1(t) - u_2(t)| \leq |B_1 - B_2| e^{Kb^2}, \;\; t \in [0,b]. \tag{27}$$

Similarly, by (15), (20) and (27),

$$
\begin{aligned}
|u_1'(t) - u_2'(t)| &\le \frac{1}{p(t)} \int_0^t p(s)|\tilde{f}(u_1(s)) - \tilde{f}(u_2(s))| \,\mathrm{d}s \\
&\le K \frac{1}{p(t)} \int_0^t p(s)|u_1(s) - u_2(s)| \,\mathrm{d}s \\
&\le Kb|B_1 - B_2|e^{Kb^2}, \ t \in [0, b].
\end{aligned}
$$

If we choose $\delta > 0$ such that

$$\delta < \frac{\epsilon}{(1 + Kb)e^{Kb^2}},$$

we get (26). □

Lemma 6 *Let u be a solution of problem* (15), (16). *Assume that there exists $a \ge 0$ such that*

$$u(t) < 0 \text{ for all } t \ge a, \quad u'(a) = 0. \tag{28}$$

Then $u'(t) > 0$ for all $t > a$ and

$$\lim_{t\to\infty} u(t) = 0, \ \lim_{t\to\infty} u'(t) = 0. \tag{29}$$

Proof. Since u fulfils (28), we have $f(u(t)) = \tilde{f}(u(t))$ for $t \in [a, \infty)$. By (9) and (28), $f(u(t)) > 0$ on $[a, \infty)$ and thus $p(t)u'(t)$ and $u'(t)$ are positive on (a, ∞). Consequently there exists $\lim_{t\to\infty} u(t) = B_1 \in (u(a), 0]$. By (15),

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) = f(u(t)), \ t > 0, \tag{30}$$

and, by multiplication and integration over $[a, t]$,

$$\frac{u'^2(t)}{2} + \int_a^t \frac{p'(s)}{p(s)} u'^2(s) \,\mathrm{d}s = F(u(a)) - F(u(t)), \ t > a. \tag{31}$$

Therefore

$$0 \le \lim_{t\to\infty} \int_a^t \frac{p'(s)}{p(s)} u'^2(s) \,\mathrm{d}s \le F(u(a)) - F(B_1) < \infty,$$

and hence $\lim_{t\to\infty} u'^2(t)$ exists. Since u is bounded on $[0, \infty)$, we get

$$\lim_{t\to\infty} u'^2(t) = \lim_{t\to\infty} u'(t) = 0.$$

By (8), (12) and (30), $\lim_{t\to\infty} u''(t)$ exists and, since u' is bounded on $[0, \infty)$, we get $\lim_{t\to\infty} u''(t) = 0$. Hence, letting $t \to \infty$ in (30), we obtain $f(B_1) = 0$. Therefore $B_1 = 0$ and (29) is proved. □

Lemma 7 *Let u be a solution of problem* (15), (16). *Assume that there exist $a_1 > 0$ and $A_1 \in (0, L)$ such that*

$$u(t) > 0 \text{ for all } t > a, \ u(a_1) = A_1, \ u'(a_1) = 0. \tag{32}$$

Then $u'(t) < 0$ for all $t > a_1$ and (29) *holds.*

Proof. Since u fulfils (32), we can find a maximal $b > a_1$ such that $0 < u(t) < L$ for $t \in [a_1, b)$ and consequently $f(u(t)) = \tilde{f}(u(t))$ for $t \in [a_1, b)$. By (9) and (32), $f(u(t)) < 0$ on $[a_1, b)$ and thus $p(t)u'(t)$ and $u'(t)$ are negative on (a_1, b). So, u is positive and decreasing on $[a_1, b)$ which yields $b = \infty$ (otherwise we get $u(b) = 0$, contrary to (32)). Consequently there exists $\lim_{t\to\infty} u(t) = L_1 \in [0, A_1)$. By multiplication and integration (30) over $[a_1, t]$, we obtain

$$\frac{u'^2(t)}{2} + \int_{a_1}^{t} \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = F(A_1) - F(u(t)), \quad t > a_1.$$

By similar argument as in the proof of Lemma 6 we get that $\lim_{t\to\infty} u'(t) = 0$ and $L_1 = 0$. Therefore (29) is proved. □

4 Damped solutions

Definition 8 A solution u of problem (15), (16) is called damped, if

$$\sup\{u(t) : t \in [0, \infty)\} < L. \tag{33}$$

Theorem 9 *If u is a damped solution of problem* (15), (16), *then u has a finite number of isolated zeros and satisfies* (29); *or u is oscillatory (it has an unbounded set of isolated zeros).*

Proof. Let u be a damped solution of problem (15), (16). According to (33) we have $\tilde{f}(u(t)) = f(u(t))$ for $t \in (0, \infty)$.
Step 1. If u has no zero in $(0, \infty)$, then $u(t) < 0$ for $t \geq 0$ and, by Lemma 6, u fulfils (29).
Step 2. Assume that $\theta > 0$ is the first zero of u on $(0, \infty)$. By (9), $u'(t) > 0$ for $t \in (0, \theta)$. Due to Remark 4, $u'(\theta) > 0$. Let $u(t) > 0$ for $t \in (\theta, \infty)$. By virtue of (9) and (15), $\tilde{f}(u(t)) < 0$ for $t \in (\theta, \infty)$ and thus $p(t)u'(t)$ is decreasing. Let u' be positive on (θ, ∞). Then u' is also decreasing, u is increasing and $\lim_{t\to\infty} u(t) = \bar{L} \in (0, L)$, due to (33). Consequently, $\lim_{t\to\infty} u'(t) = 0$. Letting $t \to \infty$ in (30), we get $\lim_{t\to\infty} u''(t) = f(\bar{L}) < 0$, which is impossible because u' is bounded from below. Therefore there are $a_1 > \theta$ and $A_1 \in (0, L)$ satisfying (32) and, by Lemma 7, either u fulfils (29) or u has the second zero $\theta_1 > a_1$ with $u'(\theta_1) < 0$. So, u is positive on (θ, θ_1) and has just one local maximum $A_1 = u(a_1)$ in (θ, θ_1). Moreover, putting $a = 0$ and $t = a_1$ in (31), we have

$$0 < \int_{0}^{a_1} \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = F(B) - F(A_1),$$

and hence

$$F(A_1) < F(B). \tag{34}$$

Step 3. Let u have no other zeros. Then $u(t) < 0$ for $t \in (\theta_1, \infty)$. Assume that u' is negative on $[\theta_1, \infty)$. Then, due to (17), $\lim_{t\to\infty} u(t) = \bar{L} \in [B, 0)$. Putting $a = a_1$ in (31) and letting $t \to \infty$, we obtain

$$0 < \lim_{t\to\infty} \left[\frac{u'^2(t)}{2} + \int_{a_1}^{t} \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s \right] = F(A_1) - F(\bar{L}).$$

Therefore $\lim_{t\to\infty} u'^2(t)$ exists and, since u is bounded, we deduce that

$$\lim_{t\to\infty} u'(t) = 0.$$

Letting $t \to \infty$ in (30), we get $\lim_{t\to\infty} u''(t) = f(\bar{L}) > 0$, which contradicts the fact that u' is bounded above. Therefore, according to (17), there exist $b_1 > \theta_1$ and $B_1 \in [B, 0)$ such that $u(b_1) = B_1$, $u'(b_1) = 0$. Then, Lemma 6 yields that u fulfils (29). Since u' is positive on (b_1, ∞), u has just one minimum $B_1 = u(b_1)$ on (θ_1, ∞). Moreover, putting $a = a_1$ and $t = b_1$ in (31), we have

$$0 < \int_{a_1}^{b_1} \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = F(A_1) - F(B_1),$$

which together with (34) yields

$$F(B_1) < F(A_1) < F(B). \tag{35}$$

Step 4. Assume that u has its third zero $\theta_2 > \theta_1$. Then we prove as in Step 2 that u has just one negative minimum $B_1 = u(b_1)$ in (θ_1, θ_2) and (35) is valid. Further, as in Step 2, we deduce that either u fulfils (29) or u has the fourth zero $\theta_3 > \theta_2$, u is positive on (θ_2, θ_3) with just one local maximum $A_2 = u(a_2) < L$ on (θ_2, θ_3), and $F(A_2) < F(B_1)$. This together with (35) yields

$$F(A_2) < F(B_1) < F(A_1) < F(B). \tag{36}$$

If u has no other zeros, we deduce as in Step 3 that u has just one negative minimum $B_2 = u(b_2)$ in (θ_3, ∞), $F(B_2) < F(A_2)$ and u fulfils (29).
Step 5. If u has other zeros, we use the previous arguments and get that either u has a finite number of zeros and then fulfils (29) or u is oscillatory. □

Remark 10 According to the proof of Theorem 9 we see, that if u is oscillatory, it has just one positive local maximum between the first and the second zero, then just one negative local minimum between the second and the third zero, and so on. By (35) and (36) and Remark 1, these maxima are decreasing (minima are increasing) for t increasing.

Lemma 11 *A solution u of problem* (15), (16) *fulfils the condition*

$$\sup\{u(t) : t \in [0, \infty)\} = L \tag{37}$$

if and only if u fulfils the condition

$$\lim_{t\to\infty} u(t) = L, \quad u'(t) > 0 \text{ for } t \in (0,\infty). \tag{38}$$

Proof. Assume that u fulfils (37). Then there exists $\theta \in (0,\infty)$ such that $u(\theta) = 0$, $u'(t) > 0$ for $t \in (0,\theta]$. Otherwise $\sup\{u(t) : t \in [0,\infty)\} = 0$, due to Lemma 6.
Let $a_1 \in (\theta,\infty)$ be such that $u'(t) > 0$ on (θ, a_1), $u'(a_1) = 0$. By Remark 4 and (37), $u(a_1) \in (0, L)$. Integrating the equality (15) over (a_1, t), we get

$$u'(t) = \frac{1}{p(t)} \int_{a_1}^{t} p(s)\tilde{f}(u(s))\, \mathrm{d}s, \quad \text{for all } t > a_1.$$

Due to (9) and (14), we see that u is strictly decreasing for $t > a_1$ as long as $u(t) \in (0, L)$. Thus, there are two possibilities. If $u(t) > 0$ for all $t > a_1$, then from Lemma 7 we get (29), which contradicts (37). If there exists $\theta_1 > a_1$ such that $u(\theta_1) = 0$, then in view Remark 4 we have $u'(\theta_1) < 0$. Using the arguments of steps 3–5 of the proof of Theorem 9, we get that u is damped, contrary to (37). Therefore such a_1 cannot exist and $u' > 0$ on $(0,\infty)$. Consequently, $\lim_{t\to\infty} u(t) = L$. So, u fulfils (38). The inverse implication is evident. □

Note that if we extend the function p in equation (15) from the half–line onto $\mathbb{R}$ (as an even function), then any solution of (15), (7) has the same limit L as $t \to -\infty$ and $t \to \infty$. Therefore we will use the following definition.

Definition 12 A solution u of problem (15), (16) satisfying (38) is called a homoclinic solution.

Theorem 13 *(On damped solutions) Let $\bar{B}$ be of* (10). *Assume that u is a solution of problem* (15), (16) *with $B \in [\bar{B}, 0)$. Then u is damped.*

Proof. Let u be a solution of (15), (16) with $B \in [\bar{B}, 0)$. Then, by (13),

$$F(B) \le F(L). \tag{39}$$

Assume on the contrary that u is not damped. Then $\sup\{u(t) : t \in [0,\infty)\} \ge L$. If $\sup\{u(t) : t \in [0,\infty)\} > L$, then there exists $b \in (0,\infty)$ such that $u(b) = L$, $u'(b) > 0$ and $u(t) < L$ for $t \in [0, b)$. Then (30) and (34) give by integration

$$0 < \frac{u'^2(b)}{2} + \int_0^b \frac{p'(s)}{p(s)} u'^2(s)\, \mathrm{d}s = F(B) - F(L) \le 0,$$

a contradiction. If $\sup\{u(t) : t \in [0,\infty)\} = L$, then, by Lemma 11, u fulfils (38). So u has a unique zero $\theta > 0$. Integrating (30) over $[0,\theta]$, we get

$$\frac{u'^2(\theta)}{2} + \int_0^\theta \frac{p'(s)}{p(s)} u'^2(s)\, \mathrm{d}s = F(B),$$

and so

$$u'^2(\theta) < 2F(B). \tag{40}$$

Integrating (30) over $[\theta, t]$, we obtain for $t > \theta$

$$\frac{u'^2(t)}{2} - \frac{u'^2(\theta)}{2} + \int_\theta^t \frac{p'(s)}{p(s)} u'^2(s)\,\mathrm{d}s = F(u(\theta)) - F(u(t)) = -F(u(t)).$$

Therefore, $u'^2(\theta) > 2F(u(t))$ on (θ, ∞), and letting $t \to \infty$, we get $u'^2(\theta) \geq 2F(L)$. This together with (40) contradicts (39). We have proved that u is damped. □

Theorem 14 *Let $\mathcal{M}_d$ be the set of all $B < 0$ such that corresponding solutions of problem* (15), (16) *are damped. Then $\mathcal{M}_d$ is open in* $(-\infty, 0)$.

Proof. Let $B_0 \in \mathcal{M}_d$ and u_0 be a solution of (15), (16) with $B = B_0$. So, u_0 is damped.
(a) Let u_0 be oscillatory. Then its first local maximum belongs to $(0, L)$. Lemma 5 guarantees that if B is sufficiently close to B_0, the corresponding solution u of (15), (16) has also its first local maximum in $(0, L)$. That means that there exist $a_1 > 0$ and $A_1 \in (0, L)$ such that u satisfies (32). Now, we can continue as in the proof of Theorem 9 using the arguments of steps 2–5 and get that u is damped.
(b) Let u_0 have at most a finite number of zeros. Then, by Theorem 9, u_0 fulfils (29). Choose $c_0 \in (0, F(L)/3)$. Since u_0 fulfils (30), we get by integration over $[0, t]$

$$\frac{u_0'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u_0'^2(s)\,\mathrm{d}s = F(B_0) - F(u_0(t)), \quad t > 0.$$

For $t \to \infty$ we get, by (29),

$$\int_0^\infty \frac{p'(s)}{p(s)} u_0'^2(s)\,\mathrm{d}s = F(B_0). \tag{41}$$

Therefore we can find $b > 0$ such that

$$\int_b^\infty \frac{p'(s)}{p(s)} u_0'^2(s)\,\mathrm{d}s < c_0. \tag{42}$$

Choose $\delta > 0$. Let $M = M(b, B_0, \delta)$ be the constant from Lemma 3. Choose $\epsilon \in (0, \frac{c_0}{2M})$. Assume that $B < 0$ and u is a corresponding solution of problem (15), (16). Using Lemma 3, Lemma 5 and the continuity of F, we can find $\bar{\delta} \in (0, \delta)$ such that if $|B - B_0| < \bar{\delta}$, then

$$|F(B) - F(B_0)| < c_0, \tag{43}$$

moreover $|u_0'(t) - u'(t)| < \epsilon$ for $t \in [0, b]$ and

$$\int_0^b \frac{p'(s)}{p(s)} |u_0'^2(s) - u'^2(s)|\,\mathrm{d}s \leq \max_{t \in [0,b]} |u_0'(t) - u'(t)| \int_0^b \frac{p'(s)}{p(s)} (|u_0'(s)| + |u'(s)|)\,\mathrm{d}s$$
$$\leq \epsilon \cdot 2M < \frac{c_0}{2M} 2M = c_0.$$

Therefore, we have

$$\int_0^b \frac{p'(s)}{p(s)} |u_0'^2(s) - u'^2(s)| \, \mathrm{d}s < c_0. \tag{44}$$

Consequently, integrating (15) over $[0, t]$ and using (41) – (44), we get for $t \geq b$

$$\begin{aligned}
F(B) - \tilde{F}(u(t)) &= \int_0^t \frac{p'(s)}{p(s)} u'^2(s) \, \mathrm{d}s + \frac{u'^2(t)}{2} \geq \int_0^t \frac{p'(s)}{p(s)} u'^2(s) \, \mathrm{d}s \\
&\geq \int_0^b \frac{p'(s)}{p(s)} u'^2(s) \, \mathrm{d}s = \int_0^b \frac{p'(s)}{p(s)} (u'^2(s) - u_0'^2(s)) \, \mathrm{d}s \\
&+ \int_0^b \frac{p'(s)}{p(s)} u_0'^2(s) \, \mathrm{d}s \geq -c_0 + \int_0^b \frac{p'(s)}{p(s)} u_0'^2(s) \, \mathrm{d}s \\
&= -c_0 + \int_0^\infty \frac{p'(s)}{p(s)} u_0'^2(s) \, \mathrm{d}s - \int_b^\infty \frac{p'(s)}{p(s)} u_0'^2(s) \, \mathrm{d}s \\
&> -c_0 + F(B_0) - c_0 = -2c_0 + F(B_0) - F(B) + F(B) \\
&> -3c_0 + F(B).
\end{aligned}$$

We get $\tilde{F}(u(t)) < 3c_0 < F(L)$ for $t \geq b$. Therefore $\tilde{F}(u(t)) = F(u(t))$ for $t \geq b$ and, due to Remark 1,

$$\sup\{u(t) : t \in [b, \infty)\} < L. \tag{45}$$

Assume that there is $b_0 \in (0, b)$ such that $u(b_0) = L$, $u'(b_0) > 0$. Then, since $(p(t)u'(t))' = 0$ if $t > b_0$ and $u(t) > L$, we get $u'(t) > 0$ and $u(t) > L$ for $t > b_0$, contrary to (45). Hence we get that u fulfils (33). □

5 Escape solutions

Definition 15 A solution u of problem (15), (16) is called escape, if there exists $c > 0$ such that

$$u(c) = L, \quad u'(t) > 0 \text{ for } t \in (0, \infty). \tag{46}$$

Theorem 16 *(On three types of solutions) Let u be a solution of problem* (15), (16). *Then u is just one of the following three types*

(I) u is damped;

(II) u is homoclinic;

(III) u is escape.

Proof. By Definition 8, u is damped if and only if (33) holds. By Lemma 11 and Definition 12, u is homoclinic if and only if (37) holds. Therefore, if u is neither damped nor homoclinic, u has to fulfil

$$\sup\{u(t) : \; t \in [0, \infty)\} > L. \tag{47}$$

Then, similarly as in the proof of Lemma 11, we get $u'(t) > 0$ for $t \in (0, \infty)$. Hence (47) is equivalent to (46). $\square$

By Theorem 13 we know that if $B \in [\bar{B}, 0)$, then a solution of problem (15), (16) is damped. Therefore if we want to prove the existence of an escape solution of (15), (16), we need to restrict our consideration on $B < \bar{B}$. First we will prove an auxiliary assertion.

Lemma 17 *Let $C < \bar{B}$ and $\{B_n\} \subset (-\infty, C)$. Then for each $n \in \mathbb{N}$*
(i) *there exists a solution u_n of problem* (15), (16) *with $B = B_n$,*
(ii) *there exists $b_n > 0$ such that $[0, b_n)$ is the maximal interval on which the solution u_n is increasing and its values are less or equal to L,*
(iii) *there exists $\gamma_n \in (0, b_n)$ satisfying $u_n(\gamma_n) = C$.*
If the sequence $\{\gamma_n\}$ is unbounded, then there exists $\ell \in \mathbb{N}$ such that u_ℓ is an escape solution.

Proof. In view of Lemma 3, Theorem 16 and the proof of Theorem 9 solutions u_n of problem (15), (16) with $B = B_n$ and constants b_n, γ_n exist (b_n can be infinite). Let $\{\gamma_n\}$ be unbounded. Then

$$\lim_{n\to\infty} \gamma_n = \infty, \quad \lim_{n\to\infty} b_n = \infty. \tag{48}$$

(Otherwise we take subsequences.) Assume on the contrary that for any $n \in \mathbb{N}$, u_n is not an escape solution. Choose $n \in \mathbb{N}$. If $b_n = \infty$, we write $u_n(b_n) = \lim_{t\to\infty} u_n(t)$ and $u_n'(b_n) = \lim_{t\to\infty} u_n'(t)$. By Theorem 16, we have

$$u_n(b_n) \in [0, L], \quad u_n'(b_n) = 0. \tag{49}$$

Consequently,

$$\tilde{f}(u_n(t)) = f(u_n(t)), \quad t \in [0, b_n). \tag{50}$$

Due to (49) there is $\bar{\gamma}_n \in [\gamma_n, b_n)$ satisfying

$$u_n'(\bar{\gamma}_n) = \max\{u_n'(t) : \ t \in [\gamma_n, b_n)\}. \tag{51}$$

By (15) and (50), u_n satisfies equation

$$u_n''(t) + \frac{p'(t)}{p(t)} u_n'(t) = f(u_n(t)), \quad t \in (0, b_n).$$

Integrating it over $[0, t]$, we get

$$\frac{u_n'^2(t)}{2} + F(u_n(t)) = F(B_n) - \int_0^t \frac{p'(s)}{p(s)} u_n'^2(s)\,\mathrm{d}s, \ t \in (0, b_n). \tag{52}$$

Put

$$E_n(t) = \frac{u_n'^2(t)}{2} + F(u_n(t)), \ t \in (0, b_n). \tag{53}$$

Then, by (52),
$$\frac{\mathrm{d}E_n(t)}{\mathrm{d}t} = -\frac{p'(t)}{p(t)} u_n'^2(t) < 0, \; t \in (0, b_n). \tag{54}$$
We see that E_n is decreasing. Since F is increasing on $[0, L]$ (see Remark 1), we get by (49) and (53),
$$E_n(\gamma_n) > F(u_n(\gamma_n)) = F(C), \; E_n(b_n) = F(u_n(b_n)) \le F(L). \tag{55}$$
Integrating (54) over (γ_n, b_n) and using (51), we obtain
$$E_n(\gamma_n) - E_n(b_n) = \int_{\gamma_n}^{b_n} \frac{p'(t)}{p(t)} u_n'^2(t)\, \mathrm{d}t \le u_n'(\bar{\gamma}_n)(L - C)K_n,$$
where
$$K_n = \sup\left\{ \frac{p'(t)}{p(t)} \; : \; t \in [\gamma_n, b_n) \right\} \in (0, \infty).$$
Further, by (55),
$$F(C) < E_n(\gamma_n) \le F(L) + u_n'(\bar{\gamma}_n)(L - C)K_n, \tag{56}$$
and
$$\frac{F(C) - F(L)}{L - C} \cdot \frac{1}{K_n} < u_n'(\bar{\gamma}_n).$$
Conditions (12) and (48) yield $\lim_{n\to\infty} K_n = 0$, which implies
$$\lim_{n\to\infty} u_n'(\bar{\gamma}_n) = \infty. \tag{57}$$
By (53) and (56),
$$\frac{u_n'^2(\bar{\gamma}_n)}{2} \le E_n(\bar{\gamma}_n) \le E_n(\gamma_n) \le F(L) + u_n'(\bar{\gamma}_n)(L - C)K_n,$$
and consequently
$$u_n'(\bar{\gamma}_n)\left(\frac{1}{2} u_n'(\bar{\gamma}_n) - (L - C)K_n\right) \le F(L) < \infty, \; n \in \mathbb{N},$$
which contradicts (57). Therefore at least one escape solution of (15), (16) with $B < \bar{B}$ must exist. □

In the next theorem we add one new assumption for the function p to our basic assumptions (8) – (12).

Theorem 18 *(On escape solutions I) Assume that the function p moreover fulfils*
$$\int_0^1 \frac{\mathrm{d}s}{p(s)} < \infty. \tag{58}$$
Then there exists $B < \bar{B}$ such that the corresponding solution of problem (15), (16) *is an escape solution.*

Proof. Let $C < \bar{B}$ and let $\{B_n\}$, $\{u_n\}$, $\{b_n\}$, and $\{\gamma_n\}$ be sequences from Lemma 17. Moreover, let $\{B_n\}$ fulfil

$$\lim_{n\to\infty} B_n = -\infty. \tag{59}$$

Assume on the contrary that for any $n \in \mathbb{N}$, u_n is not an escape solution. Then (49) is satisfied. According to (ii) from Lemma 17 and (14)

$$\tilde{f}(u_n(t)) = f(u_n(t)), \quad t \in (0, b_n), \tag{60}$$

holds. By Lemma 17, the sequence $\{\gamma_n\}$ must be bounded, that is there exists $\Gamma \in (0,\infty)$ such that

$$\gamma_n \le \Gamma \quad \text{for } n \in \mathbb{N}. \tag{61}$$

Choose an arbitrary $n \in \mathbb{N}$. Since $B_n < C < 0$ and $u_n(t) \in [B_n, C]$ for $t \in [0, \gamma_n]$, we get, by (9),

$$f(u_n(t)) > 0 \quad \text{for } t \in (0, \gamma_n].$$

By (15), (16) and (60), $p(t)u_n'(t)$ is increasing and positive on $(0, \gamma_n]$, and so

$$u_n'(t) \le p(\gamma_n)u_n'(\gamma_n)\frac{1}{p(t)}, \quad t \in (0, \gamma_n].$$

Using (61), we get by integration over $(0, \gamma_n)$

$$C - B_n \le p(\gamma_n)u_n'(\gamma_n)\int_0^\Gamma \frac{\mathrm{d}t}{p(t)}. \tag{62}$$

In view of (15) and (60) we get

$$\begin{aligned}
&\left(\frac{1}{2}p^2(t)u_n'^2(t) + p^2(t)F(u_n(t))\right)' \\
&= p(t)u_n'(t)(p'(t)u_n'(t) + p(t)u_n''(t) - p(t)f(u_n(t))) + 2p(t)p'(t)F(u_n(t)) \\
&= 2p(t)p'(t)F(u_n(t)) \ge 0
\end{aligned}$$

for each $t \in (\gamma_n, b_n)$. It follows that for $t \in (\gamma_n, b_n)$

$$\frac{1}{2}p^2(t)u_n'^2(t) + p^2(t)F(u_n(t)) \ge \frac{1}{2}p^2(\gamma_n)u_n'^2(\gamma_n) + p^2(\gamma_n)F(u_n(\gamma_n)).$$

Consequently,

$$p^2(t)u_n'^2(t) \ge p^2(\gamma_n)u_n'^2(\gamma_n) - 2p^2(t)F(C), \quad t \in (\gamma_n, b_n). \tag{63}$$

Since $\lim_{n\to\infty} B_n = -\infty$, we get by (62),

$$\lim_{n\to\infty} p(\gamma_n)u'(\gamma_n) = \infty. \tag{64}$$

Let $\{b_n\}$ be bounded, that is we can find $b_0 > 0$ such that $b_n \le b_0$ for $n \in \mathbb{N}$. Then, by (63) and (64), there is $n_0 \in \mathbb{N}$ such that for $n \ge n_0$

$$p^2(t)u_n'^2(t) > p^2(b_0), \quad t \in (\gamma_n, b_n),$$

and $u_n'^2(t) > 1$, $t \in (\gamma_n, b_n)$. Therefore $u_n'^2(b_n) \geq 1$, contrary to (49).

Let $\{b_n\}$ be unbounded. We have $\lim_{n\to\infty} b_n = \infty$ (otherwise we take a subsequence). There is $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, we have $\Gamma + 1 \leq b_n$ and, due to (63) and (64),

$$p^2(t)u_n'^2(t) > p^2(\Gamma+1)(L-C)^2, \quad t \in (\gamma_n, \Gamma+1].$$

Hence, $u_n'(t) > L - C$ for $t \in (\gamma_n, \Gamma+1]$. Integrating this we obtain

$$u_n(\Gamma+1) - u_n(\gamma_n) > (L-C)(\Gamma+1-\gamma_n).$$

Therefore, by (61), $u_n(\Gamma+1) > C + L - C = L$. Since u_n is increasing on $(0, b_n)$, we get $u_n(b_n) > L$, contrary to (49). We have proved that an escape solution must exist for some $B < \bar{B}$. □

The next theorem investigates the case when p does not fulfil (58) and we add one new assumption for f to our basic assumptions (8) – (12).

Theorem 19 *(On escape solutions II) Assume that f moreover fulfils*

$$\lim_{x\to-\infty} \frac{|x|}{f(x)} = \infty. \tag{65}$$

Then there exists $B < \bar{B}$ such that a solution of problem (15), (16) *is an escape solution.*

Proof. Let $C < \bar{B}$ and let $\{B_n\}$, $\{u_n\}$, $\{b_n\}$ and $\{\gamma_n\}$ be from Lemma 17 and let $\{B_n\}$ satisfy (59) . Assume on the contrary that for any $n \in \mathbb{N}$, u_n is not an escape solution. Then (49), (60) and (61) for some $\Gamma \in (0, \infty)$ hold. Put $P(t) = \int_0^t p(s)\,\mathrm{d}s$. By (20),

$$\lim_{t\to0^+} \frac{P(t)}{p(t)} = \lim_{t\to0^+} \varphi(t) = 0.$$

Therefore

$$\int_0^\Gamma \frac{P(t)}{p(t)}\,\mathrm{d}t = \Gamma_0 \in (0, \infty). \tag{66}$$

Choose an arbitrary $n \in \mathbb{N}$. Then u_n fulfils (15) and this yields

$$p(t)u_n'(t) = \int_0^t p(s)f(u_n(s))\,\mathrm{d}s, \quad t \in (0, \gamma_n]. \tag{67}$$

We can find $n_0 \in \mathbb{N}$ such that $B_n < 2C$ for $n \geq n_0$. Choose an arbitrary $n \geq n_0$. There exists a unique $\bar{\gamma}_n \in (0, \gamma_n)$ satisfying $u_n(\bar{\gamma}_n) = B_n/2$, and there exists $\sigma_n \in [1/2, 1]$ such that

$$f(\sigma_n B_n) = \max\{f(x) : \ x \in [B_n, B_n/2]\}.$$

By (67),

$$u_n'(t) \le f(\sigma_n B_n)\frac{P(t)}{p(t)}, \quad t \in (0, \bar{\gamma}_n],$$

and

$$u_n(\bar{\gamma}_n) - u_n(0) \le f(\sigma_n B_n)\int_0^{\bar{\gamma}_n} \frac{P(t)}{p(t)}\,\mathrm{d}t.$$

Consequently,

$$\frac{1}{2}\frac{|\sigma_n B_n|}{f(\sigma_n B_n)} \le \frac{1}{2}\frac{|B_n|}{f(\sigma_n B_n)} \le \int_0^{\bar{\gamma}_n} \frac{P(t)}{p(t)}\,\mathrm{d}t.$$

Letting $n \to \infty$ and using (61) and (65), we get $\infty \le \Gamma_0 < \infty$, a contradiction. Therefore an escape solution must exist for some $B < \bar{B}$. □

Theorem 20 *Let $\mathcal{M}_e$ be the set of all $B < 0$ such that corresponding solutions of problem* (15), (16) *are escape ones. Then $\mathcal{M}_e$ is open in* $(-\infty, 0)$.

Proof. Let $B_0 \in \mathcal{M}_e$ and u_0 be a solution of problem (15), (16) with $B = B_0$. So, u_0 is an escape solution. By Lemma 5, if $B < 0$ is sufficiently close to B_0, then the corresponding solution u of problem (15), (16) must be an escape solution, as well. □

6 Main results

Theorem 21 *(On a homoclinic solution) Let our basic assumptions* (8) – (12) *be fulfilled. Assume that moreover the assumptions of Theorem* 18 *or Theorem* 19 *be satisfied. Then problem* (6), (7) *has a strictly increasing solution with just one zero.*

Proof. Under assumptions (8) – (12), by Theorem 13 and Theorem 14, the set $\mathcal{M}_d$ is nonempty and open in $(-\infty, 0)$. By Theorem 20, the set $\mathcal{M}_e$ is open in $(-\infty, 0)$. We assume that the conditions of Theorem 18 or Theorem 19 are fulfilled. Using this theorem, we get that $\mathcal{M}_e$ is nonempty. Therefore the set $\mathcal{M}_h = (-\infty, 0) \setminus (\mathcal{M}_d \cup \mathcal{M}_e)$ is nonempty and if $B \in \mathcal{M}_h$, then the corresponding solution of problem (15), (16) is neither damped nor an escape solution. According to Theorem 16, such solution u is homoclinic. By Definition 12, u is strictly increasing on $[0, \infty)$ and fulfils (7). By Lemma 11, u satisfies (37) and so it is a solution of equation (6). □

In contrast to papers [1] and [2] we need assume neither that $f'(0)$ exists and is different from 0 nor that

$$\lim_{x \to 0^-} \frac{f(x)}{x} > 0.$$

See the following example.

Example 22 The function

$$f(x) = \begin{cases} \sqrt{|x|}\ln(|x|+1) & \text{for} \quad x < 0, \\ x(x-L) & \text{for} \quad x \in [0, L] \end{cases}$$

satisfies the conditions (8) – (10) and (65).

Acknowledgments

The authors thank the referees for valuable comments and suggestions.

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