On Nonlinear Boundary Value Problem for Systems of Differential Equations with Impulses*

IRENA RACHŮNKOVÁ, JAN TOMEČEK

Department of Mathematical Analysis and Applications of Mathematics,
Faculty of Science, Palacký University,
Tomkova 40, 779 00 Olomouc, Czech Republic
e-mail: rachunko@risc.upol.cz

Abstrakt

This work deals with nonlinear boundary value problems for systems of differential equations with impulses. Using the lower and upper functions method we prove the existence of a solution of such a problem. We consider both problems having upper functions greater than lower ones and problems with opposite ordered upper and lower functions.

Key words: First order nonlinear ordinary systems of differential equations, lower and upper functions, nonlinear boundary value conditions, impulses.

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1 Introduction.

In this paper we will study impulsive boundary value problems with nonlinear boundary conditions.

Boundary value problems with impulses have received a lot of attention in the literature. We can refer to the works [1] - [7]. Most of them deal with periodic boundary conditions. The scalar case of impulsive problems with nonlinear conditions has been studied by E. Liz in [5] and by the authors in [7]. Here, we extend the lower and upper functions method on the vector case and prove the existence results both for upper functions which are greater than lower ones $\alpha \leq \beta$ and for the opposite case of their ordering $\alpha \geq \beta$. Our proofs are based on the Schauder fixed point theorem.

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Let us consider the interval $J = [a, b] \subset \mathrm{R}$, where $a = t_0 < t_1 < \cdots < t_p < t_{p+1} = b$ and $p \in \mathrm{N}$. First, let $n \in \mathrm{N}$ and $|\cdot| : \mathrm{R}^n \to \mathrm{R}$ be the Eukleidian norm. We will work with the Banach spaces $X_j = C_n[t_j, t_{j+1}]$ (the space of functions $x_{(j)} : [t_j, t_{j+1}] \to \mathrm{R}^n$ continuous on $[t_j, t_{j+1}]$ with the norm $\|x_{(j)}\|_C = \max_{t\in[t_j,t_{j+1}]} |x_{(j)}(t)|$) for $j = 0, \dots, p$. Further with the Banach space X (the space of functions x of the following form

$$x(t) = \begin{cases} x_{(0)}(t) & \text{for} \quad t \in [a, t_1] \\ x_{(1)}(t) & \text{for} \quad t \in (t_1, t_2] \\ \dots & \\ x_{(p)}(t) & \text{for} \quad t \in (t_p, b] \end{cases}, \text{ where } x_{(j)} \in X_j \quad \text{for } j = 0, \dots, p.$$

Here we write $x = [x_{(0)}, \dots, x_{(p)}]_X$). X is endowed with the norm $\|x\| = \max_{j=0,\dots,p} \|x_{(j)}\|_C$. Similarly we will use the Banach space Y (the space of functions y of the following form

$$y(t) = \begin{cases} x_{(0)}(t) & \text{for} \quad t \in [a, t_1) \\ x_{(1)}(t) & \text{for} \quad t \in [t_1, t_2) \\ \dots & \\ x_{(p)}(t) & \text{for} \quad t \in [t_p, b] \end{cases}, \text{ where } x_{(j)} \in X_j \quad \text{for } j = 0, \dots, p,$$

with the norm $\|y\| = \max_{j=0,\dots,p} \|x_{(j)}\|_C$). We write $y = [x_{(0)}, \dots, x_{(p)}]_Y$. We say that $f : J \times \mathrm{R}^n \to \mathrm{R}^n$ fulfils the Carathéodory conditions on $J \times \mathrm{R}^n$, if f has the following properties: (i) for each $x \in \mathrm{R}^n$ the function $f(\cdot, x)$ is measurable on J; (ii) for almost each $t \in J$ the function $f(t, \cdot)$ is continuous on R^n; (iii) for each compact set $K \subset \mathrm{R}^n$ there exists Lebesgue integrable function $m_K : J \to \mathrm{R}$ such that $|f(t, x)| \le m_K(t)$ for a. e. $t \in J$ and all $x \in K$. For the set of functions satisfying the Carathéodory conditions on $J \times \mathrm{R}^n$ we write $Car(J \times \mathrm{R}^n)$. For a subset Ω of a Banach space, $\mathrm{cl}(\Omega)$ and $\partial\Omega$ stand for the closure and the boundary of Ω, respectively.

2 Existence results provided $\alpha \le \beta$.

We will investigate the impulsive problem

$$x'(t) = f(t, x(t)) \quad \text{for a. e. } t \in J, \tag{1}$$

$$x(t_j+) = I_j(x(t_j)), \quad j = 1, \dots, p, \tag{2}$$

$$h(x(a), x(b)) = 0, \tag{3}$$

where $f \in Car(J \times \mathrm{R}^n)$, $I_j \in C_n(\mathrm{R}^n)$, $j = 1, \dots, p$, and $h \in C(\mathrm{R}^{2n})$.

Definition 1. A function $F(z, x) : D \times \mathrm{R}^n \to \mathrm{R}^n$, $D \subset \mathrm{R}^m$, $m \in \mathrm{N}$, $F = (F_1, \dots, F_n)$ is called quasimonotonously nondecreasing (nonincreasing) in variable x, if for every $i = 1, \dots, n$ and for every $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in \mathrm{R}^n$ for which $x \le y$ and $x_i = y_i$, the inequality $F_i(z, x) \le F_i(z, y)$ ($F_i(z, x) \ge F_i(z, y)$) is valid for each $z \in D$.

Definition 2. By AC_n^- we mean a set of functions $x : J \to \mathrm{R}^n$, which are absolutely continuous on (t_j, t_{j+1}), $j = 0, \ldots, p$, $x(t_j) = x(t_j-)$, $j = 1, \ldots, p+1$, $u(a) = u(a+)$. A function $x \in AC_n^-$, which satisfies conditions (1) - (3) is called a solution of problem (1) - (3).

Definition 3. A function $\sigma \in AC_n^-$ is called a lower (upper) function of problem (1) - (3) provided the conditions

$$[\sigma'(t) - f(t, \sigma(t))](-1)^k \geq 0 \quad \text{for a. e. } t \in J, \tag{4}$$

$$[\sigma(t_j+) - I_j(\sigma(t_j))](-1)^k \geq 0, \quad j = 1, \ldots, p, \tag{5}$$

$$h(\sigma(a), \sigma(b))(-1)^k \geq 0, \tag{6}$$

where $k = 1$ $(k = 2)$, are satisfied.

Let α, β be lower and upper functions of problem (1) - (3) and

$$\alpha \leq \beta \text{ on } J. \tag{7}$$

Next, we define function $\gamma : J \times \mathrm{R}^n \to \mathrm{R}^n$ by

$$\gamma_i(t, x) = \begin{cases} \alpha_i(t) & \text{for} \quad x_i < \alpha_i(t), \\ x_i & \text{for} \quad \alpha_i(t) \leq x_i \leq \beta_i(t), \\ \beta_i(t) & \text{for} \quad \beta_i(t) < x_i, \end{cases} \tag{8}$$

for $t \in J$, $x = (x_1, \ldots, x_n) \in \mathrm{R}^n$, $i = 1, \ldots, n$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, and function $\tilde{f} : J \times \mathrm{R}^n \to \mathrm{R}^n$ by

$$\tilde{f}(t, x) = f(t, \gamma(t, x)), \quad f \in Car(J \times \mathrm{R}^n). \tag{9}$$

Further, we assume that

$$\left.\begin{array}{r} h(x, y) \text{ is quasimonotonously nonincreasing in variable } x, \\ \text{and nonincreasing in variable } y, \end{array}\right\} \tag{10}$$

$$\left.\begin{array}{r} I_{ji} \text{ are nondecreasing in all variables, for } j = 1, \ldots, p \\ \text{and } i = 1, \ldots, n, \qquad I_j = (I_{j1}, \ldots, I_{jn}) \end{array}\right\} \tag{11}$$

$$f(t, x) \text{ is quasimonotonously nondecreasing in variable } x \tag{12}$$

and consider an auxiliary problem

$$x'(t) = \tilde{f}(t, x(t)) \quad \text{for a. e. } t \in (t_j, t_{j+1}), \quad j = 0, \ldots, p, \tag{13}$$

$$x(t_j+) - x(t_j) = I_j(\gamma(t_j, x(t_j))) - \gamma(t_j, x(t_j)), \quad j = 1, \ldots, p, \tag{14}$$

$$x(a) = \gamma(a, x(a) - h(x(a), x(b))). \tag{15}$$

Proposition 1. *Let x be a solution of problem* (13) - (15) *and let α, β be lower and upper functions of problem* (1) - (3). *Let us suppose* (7) - (12) *hold. Then*

$$\alpha \leq x \leq \beta \text{ on } J, \tag{16}$$

and consequently x is a solution of (1) - (3).
Proof. Let us put

$$z(t) = \alpha(t) - x(t) \text{ for each } t \in J$$

(write $z = (z_1, \dots, z_n)$). Now, we take arbitrary $i \in \{1, \dots, n\}$. According to (15), we have $x_i(a) \in [\alpha_i(a), \beta_i(a)]$, which means that $z_i(a) \leq 0$.
Suppose that there exists $q_1 \in (a, t_1)$ such that

$$z_i(q_1) > 0. \tag{17}$$

Since z_i is continuous on $[a, t_1)$, we can find $q_0 \in [a, q_1)$ such that

$$z_i(q_0) = 0 \text{ and } z_i > 0 \text{ on } (q_0, q_1]. \tag{18}$$

In view of (12), (13) and the fact that $\gamma_k(t, x(t)) \geq \alpha_k(t)$ for $t \in J$ and for $k = 1, \dots, n$, $k \neq i$, we have

$$x_i'(t) = \tilde{f}_i(t, x(t)) = f_i(t, \gamma_1(t, x(t)), \dots, \alpha_i(t), \dots, \gamma_n(t, (t))) \geq f_i(t, \alpha(t))$$

for a. e. $t \in (q_0, q_1)$. According to (4), we have

$$z_i'(t) = \alpha_i'(t) - x_i'(t) \leq f_i(t, \alpha(t)) - \tilde{f}_i(t, x(t)) \leq 0, \text{ for a. e. } t \in (q_0, q_1). \tag{19}$$

Therefore

$$0 \geq \int_{q_0}^{q_1} z_i'(t)\, \mathrm{d}t = z_i(q_1) - z_i(q_0) = z_i(q_1),$$

which contradicts (17). Thus, we get

$$\alpha_i \leq x_i \text{ on } [a, t_1], \text{ for each } i = 1, \dots, n.$$

By this fact, (11) and (14), the inequalities

$$\alpha_i(t_1+) \leq I_{1i}(\alpha(t_1)) \leq I_{1i}(x(t_1)) = x_i(t_1+)$$

are true for each $i = 1, \dots, n$, so $z_i(t_1+) \leq 0$ for $i = 1, \dots, n$. Now, we take arbitrary $i \in \{1, \dots, n\}$. Let us suppose that there exists $q_1 \in (t_1, t_2]$ such that (17) is true. Then we can find $q_0 \in [t_1, q_1)$ such that

$$z_i(q_0+) = 0 \text{ and } z_i > 0 \text{ on } (q_0, q_1]. \tag{20}$$

Then, by (19), it is valid

$$0 \geq \int_{q_0}^{q_1} z_i'(t)\, \mathrm{d}t = z_i(q_1) - z_i(q_0+) = z_i(q_1)$$

and we get a contradiction to (17). We can prove the inequality $z_i \leq 0$ on $(t_1, t_2]$ similarly as in the previous paragraph. In such a way we can argue at each interval $(t_j, t_{j+1}]$, $j = 1, \dots, p$, and we get $z_i \leq 0$ on J for each $i = 1, \dots, n$.
It means that

$$\alpha \leq x \text{ on } J.$$

The second inequality in (16) can be proved similarly putting $z = x - \beta$ on J. Due to (16) we have

$$x'(t) = \tilde{f}(t, x(t)) = f(t, x(t)) \quad \text{for a. e. } t \in J,$$

and (2) is true. It remains to prove that x fulfils (3). It is sufficient to show that

$$\alpha(a) \le x(a) - h(x(a), x(b)) \le \beta(a). \tag{21}$$

Let us suppose that the first inequality in (21) is not true. Then there exists $i \in \{1, \dots, n\}$ such that

$$\alpha_i(a) > x_i(a) - h_i(x(a), x(b)).$$

In view to (15) we have $\alpha_i(a) = x_i(a)$, thus it follows from (10) that

$$0 < h_i(x(a), x(b)) \le h_i(\alpha(a), \alpha(b))$$

which contradicts (6). We prove the second inequality in (21) similarly. □

Theorem 1. *Let α, β be lower and upper functions of problem* (1) - (3). *Further, suppose that $\alpha \le \beta$ on J and* (10) - (12) *are true. Then there exists a solution x of problem* (1) - (3) *such that*

$$\alpha \le x \le \beta \text{ on } J. \tag{22}$$

Proof. Consider the integral equation

$$x(t) = \gamma(a, x(a) - h(x(a), x(b))) + \int_a^t \tilde{f}(s, x(s))\,\mathrm{d}s + \omega(t, x) \quad \text{for all } t \in J, \tag{23}$$

where

$$\omega(t, x) = \begin{cases} 0 & \text{for} \quad t \in [a, t_1] \\ I_1(\gamma(t_1, x(t_1))) - \gamma(t_1, x(t_1)) & \text{for} \quad t \in (t_1, t_2] \\ \cdots & \cdots \quad\quad \cdots \\ \sum_{j=1}^{p} [I_j(\gamma(t_j, x(t_j))) - \gamma(t_j, x(t_j))] & \text{for} \quad t \in (t_p, b]. \end{cases} \tag{24}$$

This problem is equivalent with problem (13) - (15). Further, define the number

$$M = \int_J \lambda(s)\,\mathrm{d}s + (p+1)(\|\alpha\| + \|\beta\|) + \sum_{j=1}^{p} \max\{|I_j(u)| : \alpha(t_j) \le u \le \beta(t_j)\}, \tag{25}$$

where $\lambda(t) = \sup\{|f(t, x)| : \alpha(t) \le x \le \beta(t)\}$ for all $t \in J$, and the set

$$\Omega = \{x \in X : \|x\| \le M\}.$$

Clearly Ω is nonempty, convex, closed and bounded set in X. We can check that the operator $T : \Omega \to X$ given by

$$Tx(t) = \gamma(a, x(a) - h(x(a), x(b))) + \int_a^t \tilde{f}(s, x(s))\,\mathrm{d}s + \omega(t, x),$$

where ω is defined by (24), maps Ω to itself. Now, we prove that T is continuous. Let us take sequence $\{x_m\} \subset \Omega$ and $x \in \Omega$ such that x_m converges to x in X. Then $\tilde{f}(t, x_m(t))$ converges to $\tilde{f}(t, x(t))$ for a. e. $t \in J$. Since $\{x_m\}$ is convergent in X, it follows that we can find a compact set $K \subset \mathrm{R}^n$ such that $\{x_m(t)\} \subset K$ for each $m \in N$ and $t \in J$. So there exists a function $m_K(t)$, which is Lebesgue integrable on J, with a property

$$|\tilde{f}(t, x_m(t))| \le m_K(t) \quad \text{for all } m \in N \text{ and a. e. } t \in J.$$

From the Lebesgue convergence theorem it follows that

$$\lim_{m\to\infty} \int_a^t \tilde{f}(s, x_m(s))\,\mathrm{d}s = \int_a^t \tilde{f}(s, x(s))\,\mathrm{d}s \quad \text{for each } t \in J. \tag{26}$$

Further,

$$\begin{aligned} \|Tx_m - Tx\| \le & \int_a^b |\tilde{f}(s, x_m(s)) - \tilde{f}(s, x(s))|\,\mathrm{d}s \\ & + |\gamma(a, x_m(a) - h(x_m(a), x_m(b))) - \gamma(a, x(a) - h(x(a), x(b)))| \\ & + \max_{j=0,\dots,p} \max_{t\in[t_j,t_{j+1}]} |\omega(t, x_m) - \omega(t, x)|. \end{aligned} \tag{27}$$

From (26) and continuity of functions γ, h and I_j $(j = 1, \dots, p)$, it follows that the right side of the inequality (27) approaches zero as $m \to \infty$. Therefore Tx_m converges to Tx in X.

Let us verify that $\mathrm{cl}(T(\Omega))$ is a compact set. First, we define $\Gamma_j \subset X_j$, for $j = 0, \dots, p$ by

$$\Gamma_j = \{(Tx)_{(j)} : x \in X,\ \|x\|_C \le M\},$$

where M is defined by (25). Obviously, if $y = [y_{(0)}, \dots, y_{(p)}]_X \in T(\Omega)$, then $y_{(j)} \in \Gamma_j$ for all $j = 0, \dots, p$. We can see that the functions of the set Γ_j are equicontinuous and uniformly bounded on compact interval $[t_j, t_{j+1}]$ for each $j = 0, \dots, p$. We take arbitrary sequence $\{Tx_m\} \subset T(\Omega)$ and write

$$Tx_m = [y_{m(0)}, \dots, y_{m(p)}]_X \quad \text{for each } m \in N.$$

From the Arzelà-Ascoli theorem, it follows that there exists a subsequence of $\{y_{m(0)}\} \subset \Gamma_0$ (we write $\{y_{k_m(0)}\}$), which converges in X_0 to $y_{(0)} \in X_0$. Further, by applying this theorem on sequence $\{y_{k_m(1)}\} \subset \Gamma_1$, we get $\{y_{l_m(1)}\}$, which converges to $y_{(1)} \in X_1$ in X_1. Then

$$\{y_{l_m(0)}\} \to y_{(0)} \text{ in } X_0 \text{ and } \{y_{l_m(1)}\} \to y_{(1)} \text{ in } X_1.$$

In the exactly same way, we proceed till the p-th component. Thus, there exists a subsequence of $\{Tx_m\}$, which converges in X to $[y_{(0)}, y_{(1)}, \dots, y_{(p)}]_X \in \mathrm{cl}(T(\Omega))$.

According to the Schauder fixed point theorem, there is a point $x \in \Omega$ such that

$$Tx = x,$$

which means that the function x is a solution of (23) and consequently a solution of (13) - (15). Proposition 1 implies that x is a solution of (1) - (3) and satisfies (22). □

3 Existence results provided $\alpha \geq \beta$.

Now, we will investigate the impulsive problem (1),

$$x(t_j-) = I_j(x(t_j)), \quad j = 1, \dots, p, \tag{28}$$

and (3), where $f \in Car(J \times \mathrm{R}^n)$, $I_j \in C_n(\mathrm{R}^n)$, $j = 1, \dots, p$, and $h \in C(\mathrm{R}^{2n})$.

Definition 4. By AC_n^+ we mean a set of functions $x : J \to \mathrm{R}^n$, which are absolutely continuous on (t_j, t_{j+1}), $j = 0, \dots, p$, $x(t_j) = x(t_j+)$, $j = 0, \dots, p$, $u(b) = u(b-)$. A function $x \in AC_n^+$, which satisfies conditions (1), (28), (3) is called a solution of problem (1), (28), (3).

Definition 5. A function $\sigma \in AC_n^+$ is called a lower (upper) function of problem (1), (28), (3) provided the conditions (4),

$$[I_j(\sigma(t_j)) - \sigma(t_j-)](-1)^k \geq 0, \quad j = 1, \dots, p, \tag{29}$$

and (6), where $k = 1$ $(k = 2)$, are satisfied.

Let α, β be lower and upper functions of problem (1), (28), (3) and

$$\beta \leq \alpha \text{ on } J. \tag{30}$$

Next, we define function $\gamma : J \times \mathrm{R}^n \to \mathrm{R}^n$ by

$$\gamma_i(t,x) = \begin{cases} \beta_i(t) & \text{for} \quad x_i < \beta_i(t) \\ x_i & \text{for} \quad \beta_i(t) \leq x_i \leq \alpha_i(t) \\ \alpha_i(t) & \text{for} \quad \alpha_i(t) < x_i \end{cases} \quad \text{for each } t \in J, \tag{31}$$

$x = (x_1, \dots, x_n) \in \mathrm{R}^n$, for $i = 1, \dots, n$, where $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, and function $\tilde{f} : J \times \mathrm{R}^n \to \mathrm{R}^n$ by (9).
Further, we assume that

$$\left.\begin{array}{c} h(x,y) \text{ is nondecreasing in variable } x, \\ \text{and quasimonotonously nondecreasing in variable } y, \end{array}\right\} \tag{32}$$

$$f(t,x) \text{ is quasimonotonously nonincreasing in variable } x, \tag{33}$$

it is valid (11) and consider an auxiliary problem (13),

$$x(t_j-) - x(t_j) = I_j(\gamma(t_j, x(t_j))) - \gamma(t_j, x(t_j)), \quad j = 1, \dots, p, \tag{34}$$

$$x(b) = \gamma(b, x(b) + h(x(a), x(b))). \tag{35}$$

Proposition 2. *Let x be a solution of problem* (13), (34), (35) *and let α, β be lower and upper functions of problem* (1), (28), (3). *Let us suppose* (30) - (33), (9), (11) *hold. Then*

$$\beta \leq x \leq \alpha \;\; on \; J, \tag{36}$$

and consequently x is a solution of (1), (28), (3).
Proof. Let us put

$$z(t) = x(t) - \alpha(t) \text{ for each } t \in J$$

(write $z = (z_1, \ldots, z_n)$). Now, we take arbitrary $i \in \{1, \ldots, n\}$. According to (35), we have $x_i(b) \in [\beta_i(b), \alpha_i(b)]$, which means that $z_i(b) \le 0$.
Suppose that there exists $q_1 \in [t_p, b)$ such that

$$z_i(q_1) > 0. \tag{37}$$

Since z_i is continuous on $(t_p, b]$, we can find $q_0 \in (q_1, b]$ such that

$$z_i(q_0) = 0 \text{ and } z_i > 0 \text{ on } [q_1, q_0). \tag{38}$$

In view of (33), (13) and the fact that $\gamma_k(t, x(t)) \le \alpha_k(t)$ for $t \in J$ and for $k = 1, \ldots, n$, $k \ne i$, we have

$$x_i'(t) = \tilde{f}_i(t, x(t)) = f_i(t, \gamma_1(t, x(t)), \ldots, \alpha_i(t), \ldots, \gamma_n(t, (t))) \ge f_i(t, \alpha(t))$$

for a. e. $t \in (q_1, q_0)$. According to (4), we have

$$z_i'(t) = x_i'(t) - \alpha_i'(t) \ge \tilde{f}_i(t, x(t)) - f_i(t, \alpha(t)) \ge 0, \text{ for a. e. } t \in (q_1, q_0). \tag{39}$$

Therefore

$$0 \le \int_{q_1}^{q_0} z_i'(t)\, \mathrm{d}t = z_i(q_0) - z_i(q_1) = -z_i(q_1),$$

which contradicts (37). Thus, we get

$$x_i \le \alpha_i \text{ on } [t_p, b], \text{ for each } i = 1, \ldots, n.$$

By this fact, (11) and (34), the inequalities

$$\alpha_i(t_p-) \ge I_{pi}(\alpha(t_p)) \ge I_{pi}(x(t_p)) = x_i(t_p-)$$

are true for each $i = 1, \ldots, n$, so $z_i(t_p-) \le 0$ for $i = 1, \ldots, n$. Now, we take arbitrary $i \in \{1, \ldots, n\}$. Let us suppose that there exists $q_1 \in (t_{p-1}, t_p)$ such that (37) is true. Then we can find $q_0 \in (q_1, t_p]$ such that

$$z_i(q_0-) = 0 \text{ and } z_i > 0 \text{ on } [q_1, q_0). \tag{40}$$

Then, by (39), it is valid

$$0 \le \int_{q_1}^{q_0} z_i'(t)\, \mathrm{d}t = z_i(q_0-) - z_i(q_1) = -z_i(q_1)$$

and we get a contradiction to (37). We can prove the inequality $z_i \le 0$ on $[t_{p-1}, t_p)$ similarly as in the previous paragraph. In such a way we can argue at each interval $[t_j, t_{j+1})$, $j = 0, \ldots, p-1$, and we get $z_i \le 0$ on J for each $i = 1, \ldots, n$. It means that

$$x \le \alpha \text{ on } J.$$

The second inequality in (36) can be proved similarly putting $z = \beta - x$ on J. Due to (36) we have

$$x'(t) = \tilde{f}(t, x(t)) = f(t, x(t)) \qquad \text{for a. e. } t \in J,$$

and (28) is true. It remains to prove that x fulfils (3). It is sufficient to show that

$$\beta(b) \le x(b) + h(x(a), x(b)) \le \alpha(b). \tag{41}$$

Let us suppose that the second inequality in (41) is not true. Then there exists $i \in \{1, \dots, n\}$ such that

$$\alpha_i(b) < x_i(b) + h_i(x(a), x(b)).$$

In view to (35) we have $x_i(b) = \alpha_i(b)$, thus it follows from (32) that

$$0 < h_i(x(a), x(b)) \le h_i(\alpha(a), \alpha(b))$$

which contradicts (6). We prove the second inequality in (41) similarly. □

Theorem 2. *Let α, β be lower and upper functions of problem* (1), (28), (3)*. Further, suppose that $\beta \le \alpha$ on J and* (11), (32), (33) *are true. Then there exists a solution x of problem* (1), (28), (3) *such that*

$$\beta \le x \le \alpha \ \text{on } J. \tag{42}$$

Proof. Consider the integral equation

$$x(t) = \gamma(b, x(b) + h(x(a), x(b))) + \int_b^t \tilde{f}(s, x(s))\,\mathrm{d}s + \tau(t, x) \qquad \text{for all } t \in J, \tag{43}$$

where

$$\tau(t, x) = \begin{cases} \sum\limits_{j=1}^{p} [I_j(\gamma(t_j, x(t_j))) - \gamma(t_j, x(t_j))] & \text{for} \quad t \in [a, t_1) \\ \sum\limits_{j=2}^{p} [I_j(\gamma(t_j, x(t_j))) - \gamma(t_j, x(t_j))] & \text{for} \quad t \in [t_1, t_2) \\ \cdots & \cdots \quad \cdots \\ I_p(\gamma(t_p, x(t_p))) - \gamma(t_p, x(t_p)) & \text{for} \quad t \in [t_{p-1}, t_p) \\ 0 & \text{for} \quad t \in [t_p, b] \end{cases}. \tag{44}$$

This problem is equivalent with problem (13), (34), (35). Further, define the number

$$M = \int_J \lambda(s)\,\mathrm{d}s + (p+1)(\|\alpha\| + \|\beta\|) + \sum_{j=1}^{p} \max\{|I_j(u)| : \beta(t_j) \le u \le \alpha(t_j)\}, \tag{45}$$

where $\lambda(t) = \sup\{|f(t, x)| : \beta(t) \le x \le \alpha(t)\}$ for all $t \in J$, and the set

$$\Omega = \{x \in Y : \|x\| \le M\}.$$

Clearly Ω is nonempty, convex, closed and bounded set in Y. Now, we define the operator $T : \Omega \to Y$ given by

$$Tx(t) = \gamma(b, x(b) + h(x(a), x(b))) + \int_b^t \tilde{f}(s, x(s))\,\mathrm{d}s + \tau(t, x),$$

where τ is defined by (44). Similarly, we can verify that T maps Ω to itself, is continuous and that $\mathrm{cl}(T(\Omega))$ is a compact set as in the proof of Theorem 1.

According to the Schauder fixed point theorem, there is a point $x \in \Omega$ such that

$$Tx = x,$$

which means that the function x is a solution of (43) and consequently a solution of (13), (34), (35). Proposition 2 implies that x is a solution of (1), (28), (3) and satisfies (42). □

Example 1. Let us consider problem (1) - (3), where $n = 2$, $f = (f_1, f_2)$,

$$\begin{aligned} f_1(t, x_1, x_2) &= -x_1^3 + x_2 + 10\cos t, \\ f_2(t, x_1, x_2) &= x_1 - x_2^7 + 15 + t, \end{aligned}$$

with impulsive functions I_1, I_2

$$\left.\begin{aligned} I_1(x) &= k_1 x, \quad k_1 \in (0,1), \\ I_2(x) &= k_2 x, \quad k_2 \in (0,1), \end{aligned}\right\} \tag{46}$$

and boundary value conditions (periodic conditions)

$$h(x, y) = x - y = 0. \tag{47}$$

We see that h, I_1, I_2 and f fulfil conditions (10) - (12). We can take lower and upper functions as constant functions. That is

$$\alpha = (c_1, c_2), \quad \beta = (d_1, d_2), \tag{48}$$

where $c_1, c_2, d_1, d_2 \in \mathrm{R}$ are suitable constants, such that $\alpha \le \beta$ and the conditions (4) - (6) are satisfied. We can construct such functions easily, if we take c_1, $c_2 < 0$ sufficiently small and d_1, $d_2 > 0$ sufficiently large. Thus, we can use Theorem 1 and get that problem (1) - (3) has at least one solution. On the other hand Theorem 2 cannot be used for this problem.

Example 2. Let us consider problem (1), (28), (3), where $n = 2$, $f = (f_1, f_2)$,

$$\begin{aligned} f_1(t, x_1, x_2) &= x_1^3 - x_2 + 10e^t, \\ f_2(t, x_1, x_2) &= -x_1 + x_2^5 + 15\sin t, \end{aligned}$$

with impulsive functions (46) and boundary value conditions (47). Now, conditions (11), (32) and (33) are valid. We can again take lower and upper functions in the form (48). If we take c_1, $c_2 > 0$ sufficiently large and d_1, $d_2 < 0$ sufficiently small, we get that $\beta \le \alpha$ and the conditions (4), (29), (6) are satisfied. Thus, according to Theorem 2, problem (1), (28), (3) has at least one solution. Let us note that Theorem 1 is not applicable for this problem.

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