ON NONLINEAR BOUNDARY VALUE PROBLEMS WITH IMPULSES

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Abstract. This work deals with a nonlinear boundary value problem with impulses. Such problem was investigated by many authors, see for example [1] or [2] and references therein. The main purpose of this paper is to prove the equivalence between impulsive problems and properly constructed boundary value problems without impulses. Using this equivalence we can transfer results known for boundary value problems of ordinary differential equations on boundary value problems of differential equations with impulses. Here, we use this approach and prove the lower and upper solutions method for nonlinear impulsive problems with impulsive functions having positive or negative derivatives.

1. Introduction

Let us consider the interval $J = [a, b] \subset \mathrm{R}$, where $a = t_0 < t_1 < \ldots < t_p < t_{p+1} = b$. We will work with the Banach spaces $C(J)$ (the space of functions x continuous on J with the norm $\|x\|_C = \max_{t \in J} |x(t)|$), $C^1(J)$ (the space of functions x having continuous first derivatives on J with the norm $\|x\|_{C^1} = \|x\|_C + \|x'\|_C$), $L(J)$ (the space of functions y Lebesgue integrable on J with the norm $\|y\|_1 = \int_a^b |y(t)| dt$) and with the set $AC(J)$ (the set of functions absolutely continuous on J). We say that $f : J \times \mathrm{R} \to \mathrm{R}$ fulfils the Carathéodory conditions on $J \times \mathrm{R}$, if f has the following properties: (i) for each $x \in \mathrm{R}$ the function $f(\cdot, x)$ is measurable on J; (ii) for almost each $t \in J$ the function $f(t, \cdot)$ is continuous on R; (iii) for each compact set $K \subset \mathrm{R}$ the function $m_K(t) = \sup_{x \in K} |f(t, x)|$ is Lebesgue integrable on J. For the set of functions satisfying the Carathéodory conditions on $J \times \mathrm{R}$ we write $Car(J \times \mathrm{R})$. For a subset Ω of a Banach space, $\mathrm{cl}(\Omega)$ and $\partial \Omega$ stand for the closure and the boundary of Ω, respectively.

We will investigate the impulsive problem

$$u'(t) = f(t, u(t)) \quad \text{for a. e. } t \in (t_j, t_{j+1}), \quad j = 0, \ldots, p, \tag{1}$$

$$u(t_j+) = I_j(u(t_j)), \quad j = 1, \ldots, p, \tag{2}$$

$$h(u(a), u(b)) = 0, \tag{3}$$

where $f \in Car(J \times \mathrm{R})$, $I_j \in C^1(\mathrm{R})$, $I_j' \neq 0$, $j = 1, \ldots, p$, and $h \in C(\mathrm{R}^2)$.

Together with the problem (1) - (3) we will study the problem without impulses

$$x'(t) = g(t, x(t)) \quad \text{for a. e. } t \in J, \tag{4}$$

$$h(x(a), w(x(b))) = 0, \tag{5}$$

where $g \in Car(J \times \mathrm{R})$, $w \in C^1(\mathrm{R})$, $w' \neq 0$, and $h \in C(\mathrm{R}^2)$.

We prove that problems (1) - (3) and (4), (5) are equivalent and by means of this we get the lower and upper functions method as well as the existence of solutions to problem (1) - (3). Let us note that the existence results to problem (1) - (3) were proven also by E. Liz in [2], but by another approach and for increasing impulsive functions, only. Here, we extend the lower and upper functions method and the existence results on the case of decreasing impulsive functions. Our proofs need no technique and results from the theory of impulsive differential equations.

Definition 1. By AC^* we mean a set of functions $u : J \to \mathrm{R}$, which are absolutely continuous on each (t_i, t_{i+1}), $i = 0, \ldots, p$, $u(t_j) = u(t_j-)$, $j = 1, \ldots, p+1$, $u(a) = u(a+)$. A function $u \in AC^*$ which satisfies conditions (1) - (3) is called a solution of problem (1) - (3). A function $x \in AC(J)$ which satisfies conditions (4), (5) is called a solution of problem (4), (5).

2. Nonlinear boundary value problems without impulses

To have this paper self-contained, let us show the ideas of the lower and upper functions method for problem (4), (5).

Definition 2. A function $\alpha_1 \in AC^*$ ($\alpha_2 \in AC^*$) is called a lower (upper) function of problem (4), (5) provided the conditions

$$(\alpha_i'(t) - g(t, \alpha_i(t)))(-1)^i \geq 0 \quad \text{for a. e. } t \in (t_j, t_{j+1}), \quad j = 0, \ldots, p, \tag{6}$$

$$(\alpha_i(t_j+) - \alpha_i(t_j))(-1)^i \geq 0, \quad j = 1, \ldots, p, \tag{7}$$

$$h(\alpha_i(a), w(\alpha_i(b)))(-1)^i \geq 0, \quad i = 1, 2 \tag{8}$$

are satisfied.

We will assume certain relation between lower and upper functions.
First, let

$$\alpha_1(t) \leq \alpha_2(t) \text{ for each } t \in J, \tag{9}$$

$$\alpha(t, x) = \begin{cases} \alpha_1(t) & \text{if} \quad x < \alpha_1(t) \\ x & \text{if} \quad \alpha_1(t) \leq x \leq \alpha_2(t) \\ \alpha_2(t) & \text{if} \quad \alpha_2(t) < x \end{cases} \quad \text{for each } t \in J \text{ and } x \in \mathrm{R}, \tag{10}$$

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$$\tilde{g}(t, x) = g(t, \alpha(t, x)), \quad g \in Car(J \times \mathrm{R}). \tag{11}$$

Further, we assume that

$$w \in C^1(\mathrm{R}), \ w' > 0 \text{ on } \mathrm{R}, \tag{12}$$

$$h \in C(\mathrm{R}^2), \ h \text{ is nonincreasing in its second variable} \tag{13}$$

and consider an auxiliary problem

$$x'(t) = \tilde{g}(t, x(t)) \text{ for a. e. } t \in J, \tag{14}$$

$$x(a) = \alpha(a, x(a) - h(x(a), w(x(b)))). \tag{15}$$

Proposition 1. *Let us suppose (9) - (13) hold. Let x be a solution of problem (14), (15) and let α_1, α_2 be lower and upper functions of problem (4), (5). Then*

$$\alpha_1 \le x \le \alpha_2 \text{ on } J \tag{16}$$

and consequently x is a solution of (4), (5), as well.

Proof. Let us put

$$z(t) = \alpha_1(t) - x(t) \text{ for each } t \in J.$$

According to (15) we have $x(a) \in [\alpha_1(a), \alpha_2(a)]$, which means that $z(a) \le 0$. Suppose that there exists $q_1 \in (a, t_1)$ such that

$$z(q_1) > 0. \tag{17}$$

Since z is continuous on $[a, t_1)$, we can find $q_0 \in [a, q_1)$ such that

$$z(q_0) = 0 \text{ and } z > 0 \text{ on } (q_0, q_1]. \tag{18}$$

In view of (6) we have

$$z'(t) = \alpha_1'(t) - x'(t) \le g(t, \alpha_1(t)) - \tilde{g}(t, x(t)) = 0$$

for a. e. $t \in (q_0, q_1]$. Therefore

$$0 \ge \int_{q_0}^{q_1} z'(t)dt = z(q_1) - z(q_0) = z(q_1),$$

which contradicts (17). Thus we get

$$z \le 0 \text{ on } [a, t_1]. \tag{19}$$

By (7) and (19), the inequalities $\alpha_1(t_1+) \le \alpha_1(t_1) \le x(t_1) = x(t_1+)$ are true, and so $z(t_1+) \le 0$.

Suppose that there exists $q_1 \in (t_1, t_2)$ such that (17) is true. Then we can find $q_0 \in (t_1, q_1)$

such that (18) is valid and we get a contradiction to (17) as before. In such a way we can argue at each interval $(t_j, t_{j+1}]$, $j = 1, \ldots, p$, and get $z \leq 0$ on J which means that

$$\alpha_1 \leq x \text{ on } J.$$

The second inequality in (16) can be proved similarly putting $z = x - \alpha_2$ on J.
Due to (16), we have

$$x'(t) = \tilde{g}(t, x(t)) = g(t, x(t)) \text{ for a. e. } t \in J.$$

It remains to prove that x fulfils (5). It is sufficient to show that

$$\alpha_1(a) \leq x(a) - h(x(a), w(x(b))) \leq \alpha_2(a), \tag{20}$$

because (15) implies (5) then.
Suppose on the contrary, that

$$\alpha_1(a) > x(a) - h(x(a), w(x(b))).$$

Since (15) is valid, we get $x(a) = \alpha_1(a)$. Then, by (12) and (13), we have $0 < h(x(a), w(x(b))) \leq h(\alpha_1(a), w(\alpha_1(b)))$, which contradicts (8). The second inequality in (20) can be proved similarly. This completes the proof. □

We can easily see that the following modification of Proposition 1 is true.

Proposition 2. *The assertion of Proposition 1 remains valid if we replace the conditions (12) and (13) with*

$$w \in C^1(\mathrm{R}), \ w' < 0 \ \textit{on } \mathrm{R} \tag{21}$$

and

$$h \in C(\mathrm{R}^2), \ h \textit{ is nondecreasing in its second variable.} \tag{22}$$

Theorem 1. *Let α_1 and α_2 be lower and upper functions of problem (4), (5). Further, suppose that $\alpha_1 \leq \alpha_2$ on J and that either $w' > 0$ and h is nonincreasing in its second variable or $w' < 0$ and h is nondecreasing in its second variable. Then there exists a solution x of problem (4), (5) such that*

$$\alpha_1 \leq x \leq \alpha_2 \ \textit{on } J. \tag{23}$$

Proof. Consider the integral equation

$$x(t) = \alpha(a, x(a) - h(x(a), w(x(b)))) + \int_a^t \tilde{g}(s, x(s))ds, \tag{24}$$

which is equivalent with problem (14), (15). Further, define the set

$$\Omega = \{x \in C(J) : \|x\|_C \leq M\},$$

where $M = \sup_{t \in J} |\alpha_1(t)| + \sup_{t \in J} |\alpha_2(t)| + \int_a^b \lambda(s)ds$ and $\lambda(t) = \sup\{|g(t,x)| : x \in [\alpha_1(t), \alpha_2(t)]\}$. Clearly Ω is nonempty, convex, closed and bounded set in $C(J)$. We can check that the operator $T : \Omega \to C(J)$ given by

$$(Tx)(t) = \alpha(a, x(a) - h(x(a), w(x(b)))) + \int_a^t \tilde{g}(s, x(s))ds$$

is continuous and that $\mathrm{cl}(T(\Omega))$ is a compact set in $C(J)$ and T maps Ω to itself. Thus, according to the Schauder fixed point theorem, there is a point $x \in \Omega$ such that

$$Tx = x,$$

which means that the function x is a solution of (24) and consequently a solution of (14), (15). Propositions 1 and 2 imply that x is a solution of (4), (5) and satisfies (23). $\square$

Now, suppose
$$\alpha_2(t) \leq \alpha_1(t) \text{ for each } t \in J, \tag{25}$$

$$\alpha(t,x) = \begin{cases} \alpha_2(t) & \text{if} \quad x < \alpha_2(t) \\ x & \text{if} \quad \alpha_2(t) \leq x \leq \alpha_1(t) \\ \alpha_1(t) & \text{if} \quad \alpha_1(t) < x \end{cases} \text{ for each } t \in J \text{ and } x \in \mathrm{R} \tag{26}$$

and consider an auxiliary problem (14),

$$x(b) = \alpha(b, x(b) + h(x(a), w(x(b)))). \tag{27}$$

Proposition 3. *Let the conditions (25), (26), (11) and*

$$h \in C(\mathrm{R}^2), \ h \text{ is nondecreasing in its first variable, } w \in C^1(\mathrm{R}) \tag{28}$$

be satisfied. Let x be a solution of problem (14), (27) and let α_1, α_2 be lower and upper functions of problem (4), (5). Then

$$\alpha_2 \leq x \leq \alpha_1 \text{ on } J \tag{29}$$

and consequently x is a solution of (4), (5), as well.
Proof. This proof is similar to that of Proposition 1. We put

$$z(t) = x(t) - \alpha_1(t) \text{ for all } t \in J$$

and prove $z(t) \leq 0$ for all $t \in J$. In view of (27) we have $x(b) \in [\alpha_2(b), \alpha_1(b)]$, which means that $z(b) \leq 0$. Suppose, that there is $q_0 \in (t_p, b)$ such that

$$z(q_0) > 0. \tag{30}$$

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Since z is continuous on $(t_p, b]$, we can find $q_1 \in (q_0, b]$ such that

$$z(q_1) = 0 \text{ and } z > 0 \text{ on } [q_0, q_1). \tag{31}$$

In view of (6) we have

$$z'(t) = x'(t) - \alpha_1'(t) \geq \tilde{g}(t, x(t)) - g(t, \alpha_1(t)) = 0 \text{ for a. e. } t \in [q_0, q_1).$$

Therefore

$$0 \leq \int_{q_0}^{q_1} z'(t) dt = z(q_1) - z(q_0) = -z(q_0),$$

which contradicts (30). Similarly, we can proceed to the initial point $a = t_0$. To prove the first inequality in (29) we work with

$$z(t) = \alpha_2(t) - x(t) \text{ for each } t \in J.$$

Due to (29), we have

$$x'(t) = \tilde{g}(t, x(t)) = g(t, x(t)) \text{ for a. e. } t \in J.$$

Finally, we need to prove

$$\alpha_2(b) \leq x(b) + h(x(a), w(x(b))) \leq \alpha_1(b). \tag{32}$$

If $\alpha_1(b) < x(b) + h(x(a), w(x(b)))$, then $x(b) = \alpha_1(b)$ and

$$0 < h(x(a), w(x(b))) \leq h(\alpha_1(a), w(\alpha_1(b))),$$

which contradicts (8). For the first inequality in (32) we argue similarly. $\square$

Theorem 2. *Let α_1 and α_2 be lower and upper functions of problem (4), (5). Further, suppose that $\alpha_1 \geq \alpha_2$ on J and that h is nondecreasing in its first variable. Then there exists a solution x of problem (4), (5) such that*

$$\alpha_2 \leq x \leq \alpha_1 \text{ on } J. \tag{33}$$

Proof. Starting with the integral equation

$$x(t) = \alpha(b, x(b) + h(x(a), w(x(b)))) + \int_b^t \tilde{g}(s, x(s)) ds$$

and using Proposition 3 instead of Propositions 1, 2, we can argue as in the proof of Theorem 1. $\square$

3. Relation between impulsive problem (1) - (3) and problem (4), (5)

Consider the impulsive functions $I_j \in C^1(\mathrm{R})$, $I_j' \neq 0$, $j = 1, \ldots, p$, and define functions $w_i : \mathrm{R} \to \mathrm{R}$, $i = 0, \ldots, p$:

$$
\begin{aligned}
w_0 &= \mathrm{id}_{\mathrm{R}}, \\
w_1 &= I_1, \\
w_2 &= I_2(I_1) = I_2(w_1), \\
&\ldots \\
w_{p-1} &= I_{p-1}(w_{p-2}), \\
w &= w_p = I_p(w_{p-1}).
\end{aligned}
\tag{34}
$$

Further, having $f \in Car(J \times \mathrm{R})$, define

$$
g(t,x) = \begin{cases}
\frac{f(t, w_0(x))}{w_0'(x)} & \text{for} \quad t \in [a, t_1] \\
\frac{f(t, w_1(x))}{w_1'(x)} & \text{for} \quad t \in (t_1, t_2] \\
\quad \ldots \\
\frac{f(t, w_p(x))}{w_p'(x)} & \text{for} \quad t \in (t_p, b]
\end{cases}
\quad \text{for } x \in \mathrm{R}.
\tag{35}
$$

Conversely, having $g \in Car(J \times \mathrm{R})$, define

$$
f(t,u) = \begin{cases}
g(t, w_0^{-1}(u)) w_0'(w_0^{-1}(u)) & \text{for} \quad t \in [a, t_1] \\
g(t, w_1^{-1}(u)) w_1'(w_1^{-1}(u)) & \text{for} \quad t \in (t_1, t_2] \\
\quad \ldots \\
g(t, w_p^{-1}(u)) w_p'(w_p^{-1}(u)) & \text{for} \quad t \in (t_p, b]
\end{cases}
\quad \text{for } u \in \mathrm{R}.
\tag{36}
$$

Theorem 3. *a) Let u be a solution of problem (1) - (3) and let g be defined by (34), (35). Then a function x given by*

$$
x(t) = \begin{cases}
w_0^{-1}(u(t)) & \text{for} \quad t \in [a, t_1] \\
w_1^{-1}(u(t)) & \text{for} \quad t \in (t_1, t_2] \\
\quad \ldots \\
w_p^{-1}(u(t)) & \text{for} \quad t \in (t_p, b],
\end{cases}
\tag{37}
$$

is a solution of problem (4), (5).
b) Let x be a solution of problem (4), (5) and let f be defined by (34), (36). Then a function u given by

$$
u(t) = \begin{cases}
w_0(x(t)) & \text{for} \quad t \in [a, t_1] \\
w_1(x(t)) & \text{for} \quad t \in (t_1, t_2] \\
\quad \ldots \\
w_p(x(t)) & \text{for} \quad t \in (t_p, b],
\end{cases}
\tag{38}
$$

is a solution of problem (1) - (3).
Proof. a) Let u be a solution of problem (1) - (3). Then, in view of (37) and (35)

$$
x'(t) = \frac{u'(t)}{w_i'(x(t))} = \frac{f(t, w_i(x(t)))}{w_i'(x(t))} = g(t, x(t)) \text{ for a. e. } t \in (t_i, t_{i+1}),
$$

$i = 0, \ldots, p$. Since $u(a) = x(a)$ and $u(b) = w_p(x(b)) = w(x(b))$, we have by (3), $h(x(a), w(x(b))) = 0$.

Next, $x(t_j+) = w_j^{-1}(u(t_j+)) = w_j^{-1}(I_j(u(t_j))) = w_{j-1}^{-1}(I_j^{-1}(I_j(u(t_j)))) = w_{j-1}^{-1}(u(t_j)) = x(t_j)$, $j = 1, \ldots, p$. This together with $u \in AC^*$ implies $x \in AC(J)$ and therefore x is a solution of (4), (5).

b) Let x be a solution of (4), (5). Then, by (38) and (36), $u'(t) = w_i'(x(t))x'(t) = w_i'(x(t))g(t, x(t)) = w_i'(w_i^{-1}(u(t)))g(t, w_i^{-1}(u(t))) = f(t, u(t))$ for a. e. $t \in (t_i, t_{i+1})$, $i = 0, \ldots, p$.

We have $x(a) = u(a)$, $w(x(b)) = w_p(x(b)) = u(b)$ and then $h(u(a), u(b)) = 0$. Finally, $u(t_j+) = w_j(x(t_j+)) = w_j(x(t_j)) = I_j(w_{j-1}(x(t_j))) = I_j(u(t_j))$, $j = 1, \ldots, p$.

Since $x \in AC(J)$, then, in view of (38), $w_i(x(t))$ are absolutely continuous on (t_i, t_{i+1}), $i = 0, \ldots, p$ and $u(t_j-) = u(t_j)$, $j = 1, \ldots, p+1$. Therefore u is a solution of (1) - (3). $\square$

4. Lower and upper functions method to problem (1) - (3)

4.1 Increasing impulsive functions

Let us suppose

$$I_j \in C^1(\mathrm{R}), \ I_j' > 0, \quad j = 1, \ldots, p. \tag{39}$$

Definition 3. A function $\sigma_1 \in AC^*$ ($\sigma_2 \in AC^*$) is called a lower (upper) function of problem (1) - (3), provided the conditions

$$(\sigma_i'(t) - f(t, \sigma_i(t)))(-1)^i \geq 0 \ \text{ for a. e. } t \in (t_j, t_{j+1}), \quad j = 0, \ldots, p, \tag{40}$$

$$(\sigma_i(t_j+) - I_j(\sigma_i(t_j)))(-1)^i \geq 0, \quad j = 1, \ldots, p, \tag{41}$$

$$h(\sigma_i(a), \sigma_i(b))(-1)^i \geq 0, \quad i = 1, 2 \tag{42}$$

are satisfied.

Theorem 4. *Suppose that (39) holds.*

a) Let σ_1 and σ_2 be lower and upper functions of problem (1) - (3) and let g be defined by (34), (35). Then functions α_1 and α_2 given by

$$\alpha_i(t) = \begin{cases} w_0^{-1}(\sigma_i(t)) & for \quad t \in [a, t_1] \\ w_1^{-1}(\sigma_i(t)) & for \quad t \in (t_1, t_2] \\ \quad \ldots \\ w_p^{-1}(\sigma_i(t)) & for \quad t \in (t_p, b]. \end{cases} \tag{43}$$

are lower and upper functions of problem (4), (5).

b) Let α_1 and α_2 be lower and upper functions of problem (4), (5) and let f be defined by (34), (36).

Then functions σ_1 and σ_2 given by

$$\sigma_i(t) = \begin{cases} w_0(\alpha_i(t)) & for \quad t \in [a, t_1] \\ w_1(\alpha_i(t)) & for \quad t \in (t_1, t_2] \\ \quad \ldots \\ w_p(\alpha_i(t)) & for \quad t \in (t_p, b], \end{cases} \tag{44}$$

are lower and upper functions of problem (1) - (3).

Proof. We will prove the assertion for lower functions, only. Choose arbitrary $j \in \{1, \ldots, p\}$ and $i \in \{0, \ldots, p\}$. Let σ_1 be a lower function of (1) - (3). Then

$$\alpha_1'(t) = \frac{\sigma_1'(t)}{w_i'(\alpha_1(t))} \leq \frac{f(t, \sigma_1(t))}{w_i'(\alpha_1(t))} = g(t, \alpha_1(t)) \text{ for a. e. } t \in (t_i, t_{i+1}).$$

Since $\sigma_1(a) = \alpha_1(a)$ and $\sigma_1(b) = w_p(\alpha_1(b)) = w(\alpha_1(b))$, we have

$$h(\alpha_1(a), w(\alpha_1(b))) = h(\sigma_1(a), \sigma_1(b)) \leq 0.$$

Further, $\alpha_1(t_j+) = w_j^{-1}(\sigma_1(t_j+)) \leq w_j^{-1}(I_j(\sigma_1(t_j))) = w_{j-1}^{-1}(\sigma_1(t_j)) = \alpha_1(t_j)$.
Thus α_1 is a lower function of (4), (5).
Now, let α_1 be a lower function of (4), (5). Then $\sigma_1'(t) = w_i'(\alpha_1(t))\alpha_1'(t) \leq w_i'(\alpha_1(t))g(t, \alpha_1(t)) = w_i'(w_i^{-1}(\sigma_1(t)))g(t, w_i^{-1}(\sigma_1(t))) = f(t, \sigma_1(t))$, for a. e. $t \in (t_i, t_{i+1})$.
As before, we have

$$h(\sigma_1(a), \sigma_1(b)) = h(\alpha_1(a), w(\alpha_1(b))) \leq 0.$$

Finally, $\sigma_1(t_j+) = w_j(\alpha_1(t_j+)) \leq w_j(\alpha_1(t_j)) = I_j(w_{j-1}(\alpha_1(t_j))) = I_j(\sigma_1(t_j))$.
So, σ_1 is a lower function of (1) - (3). $\square$

Theorem 5. *Let σ_1 and σ_2 be lower and upper functions of problem (1) - (3) and let (39) be fulfilled. Further, let one of the following conditions be satisfied:*

1. *$\sigma_1 \leq \sigma_2$ on J and h is nonincreasing in its second variable;*

2. *$\sigma_2 \leq \sigma_1$ on J and h is nondecreasing in its first variable.*

Then there exists a solution of problem (1) - (3) lying between σ_1 and σ_2 on J.

Proof. We will investigate only the case $\sigma_1 \leq \sigma_2$ on J. The second case can be proved similarly. Consider problem (4), (5), where g is defined by (34), (35). We can see that $f \in Car(J \times \mathrm{R})$ implies $g \in Car(J \times \mathrm{R})$, and that, according to (39), we get $w \in C^1(\mathrm{R})$, $w' > 0$. Theorem 4 implies that the functions α_1, α_2 given by (43) are lower and upper functions of (4), (5).
Since the functions w_i, $i = 0, \ldots, p$, are increasing, it follows, that $\alpha_1 \leq \alpha_2$ on J if and only if $\sigma_1 \leq \sigma_2$ on J. Therefore, by Theorem 1, problem (4), (5) has a solution x which satisfies (23). Thus, in view of Theorem 3, the function u given by (38) is a solution of (1) - (3) and $\sigma_1 \leq u \leq \sigma_2$ on J. $\square$

4.2 Decreasing impulsive function

Let us suppose

$$I_j \in C^1(\mathrm{R}), \ I_j' < 0, \quad j = 1, \ldots, p. \tag{45}$$

Definition 4. A function $\sigma_1 \in AC^*$ ($\sigma_2 \in AC^*$) is called a generalized lower (upper) function of problem (1) - (3), provided the conditions

$$(\sigma_i'(t) - f(t, \sigma_i(t)))(-1)^{i+j} \geq 0 \quad \text{for a. e. } t \in (t_j, t_{j+1}), \quad j = 0, \ldots, p, \tag{46}$$

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$$(\sigma_i(t_j+) - I_j(\sigma_i(t_j)))(-1)^{i+j} \geq 0, \quad j = 1, \ldots, p, \tag{47}$$

$$h(\sigma_i(a), \sigma_i(b))(-1)^i \geq 0, \quad i = 1, 2. \tag{48}$$

Theorem 6. *Suppose that (45) is satisfied.*
a) Let σ_1 and σ_2 be generalized lower and upper functions of problem (1) - (3) and let g be defined by (34), (35). Then α_1, α_2 given by (43) are lower and upper function of (4), (5).
b) Let α_1, α_2 be lower and upper functions of (4), (5) and let f be defined by (34), (36). Then σ_1, σ_2 given by (44) are generalized lower and upper functions of problem (1) - (3).
Proof. We can use the arguments from the proof of Theorem 4 having in mind that the conditions (34) and (45) imply that

$$(-1)^j w_j' > 0, \ j = 0, \ldots, p. \tag{49}$$

We will prove the assertion a) for σ_2, only. So, choose arbitrary $j \in \{0, \ldots, p\}$ and consider a generalized upper function σ_2 of (1) - (3). Then

$$\alpha_2'(t) - g(t, \alpha_2(t)) = \frac{\sigma_2'(t)}{w_j'(\alpha_2(t))} - \frac{f(t, \sigma_2(t))}{w_j'(\alpha_2(t))} = (\sigma_2'(t) - f(t, \sigma_2(t)))\frac{(-1)^j}{|w_j'(\alpha_2(t))|} \geq 0$$

for a. e. $t \in (t_j, t_{j+1})$.
Further, choose $j \in \{1, \ldots, p\}$ and suppose that j is odd. Then, by (47), $\sigma_2(t_j+) \leq I_j(\sigma_2(t_j))$, and in view of (49) we have

$$w_j^{-1}(\sigma_2(t_j+)) \geq w_j^{-1}(I_j(\sigma_2(t_j))).$$

Therefore, $\alpha_2(t_j+) = w_j^{-1}(\sigma_2(t_j+)) \geq w_j^{-1}(I_j(\sigma_2(t_j))) = w_{j-1}^{-1}(\sigma_2(t_j)) = \alpha_2(t_j)$. Similarly for j even. As in the proof of Theorem 4, we get $h(\alpha_2(a), w(\alpha_2(b))) \geq 0$. $\square$

Theorem 7. *Let σ_1 and σ_2 be generalized lower and upper functions of problem (1) - (3) and let (45) be fulfilled. Further, let one of the following conditions be satisfied:*

1. $(\sigma_1 - \sigma_2)(-1)^i \leq 0$ *on $(t_i, t_{i+1}]$ for $i = 0, \ldots, p$ and h is nonincreasing (nondecreasing) in its second variable if p is even (odd);*

2. $(\sigma_1 - \sigma_2)(-1)^i \geq 0$ *on $(t_i, t_{i+1}]$ for $i = 0, \ldots, p$ and h is nondecreasing in its first variable.*

Then there exists a solution of problem (1) - (3) lying between σ_1 and σ_2 on J.
Proof. We follow the proof of the Theorem 5 working with $(-1)^i w_i$, $i = 0, \ldots, p$ and using Theorem 6 instead of Theorem 4. $\square$

We can also study instead of problem (4), (5) the problem (4),

$$h(w(x(a)), x(b)) = 0, \tag{50}$$

where $g \in Car(J \times \mathrm{R})$ is given by

$$g(t,x) = \begin{cases} \frac{f(t,w_0(x))}{w_0'(x)} & \text{for} \quad t \in (t_p, b] \\ \frac{f(t,w_1(x))}{w_1'(x)} & \text{for} \quad t \in (t_{p-1}, t_p] \\ \quad \dots & \\ \frac{f(t,w_p(x))}{w_p'(x)} & \text{for} \quad t \in [a, t_1] \end{cases} \quad \text{for } x \in \mathrm{R}, \tag{51}$$

where

$$\begin{aligned} w_0 &= \mathrm{id}_{\mathrm{R}}, \\ w_1 &= I_1^{-1}, \\ w_2 &= I_2^{-1}(I_1^{-1}) = I_2^{-1}(w_1), \\ &\dots \\ w_{p-1} &= I_{p-1}^{-1}(w_{p-2}), \\ w &= w_p = I_p^{-1}(w_{p-1}). \end{aligned} \tag{52}$$

Working with problem (4), (50) we can prove the following modification of Theorem 7.

Theorem 8. *Let σ_1 and σ_2 be generalized lower and upper function of problem (1) - (3) and let (45) be fulfilled. Further, let one of the following conditions be satisfied:*

1. *$(\sigma_1 - \sigma_2)(-1)^{p-i} \leq 0$ on $(t_i, t_{i+1}]$ for $i = 0, \dots, p$ and h is nonincreasing in its second variable;*

2. *$(\sigma_1 - \sigma_2)(-1)^{p-i} \geq 0$ on $(t_i, t_{i+1}]$ for $i = 0, \dots, p$ and h is nondecreasing (nonincreasing) in its first variable if p is even (odd).*

Then there exists a solution of problem (1) - (3) lying between σ_1 and σ_2 on J.

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