Singular Dirichlet Problem for Ordinary Differential Equation with Impulses*

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Abstract

The paper deals with the impulsive Dirichlet problem

$$u''(t) = f(t, u(t), u'(t)),$$

$$u(0) = A, \quad u(T) = B,$$

$$u(t_j+) = I_j(u(t_j)), \quad u'(t_j+) = M_j(u'(t_j)), \quad j = 1, \dots, p,$$

where $f \in Car((0,T) \times \mathbb{R}^2)$, f has time singularities at $t = 0$ and $t = T$, I_j, $M_j \in C^0(\mathbb{R})$, A, $B \in \mathbb{R}$. We prove the existence of a solution to this problem under the assumption that there exist lower and upper functions associated with the problem. Our proofs are based on the Schauder fixed point theorem and on the method of a priori estimates.

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Key words: Singular ordinary differential equation of the second order, lower and upper functions, Dirichlet conditions, impulses, time singularities.

1 Introduction

The theory of impulsive differential equations is a lot richer than the corresponding theory of differential equations (see e. g. [30] or [22]). Moreover, impulsive equations seem to represent a natural framework for mathematical modelling of several real world phenomena. It is known that bursting rhytm models in medicine and biology, optimal control models in economics, pharmacokinetics and frequency modulated systems do exhibit impulsive effects ([6], [7], [14], [15], [17]). Kruger–Thiemer model for drug distribution ([21]) is a nice illustration of simple impulsive problem. On the other hand, in certain problems in fluid dynamics and boundary layer theory ([9], [10]), the generalized Emden–Fowler

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equation $u'' + \psi(t)u^\lambda = 0$ arised. Here, ψ is continuous on $(0,1)$ but it is not Lebesgue integrable on $[0,1]$. It means that ψ has time singularities at $t = 0$ and $t = 1$ and we call such equation singular.

Motivated by these examples we will consider A, $B \in \mathbb{R}$, $T > 0$, $[0,T] \subset \mathbb{R}$, a division D of the interval $[0,T]$,

$$D = \{t_1, \dots, t_p\}, \quad p \in \mathbb{N}$$

such that $0 = t_0 < t_1 < \ldots < t_p < t_{p+1} = T$ and the problem

$$u''(t) + f(t, u(t), u'(t)) = 0, \tag{1}$$

$$u(0) = A, \quad u(T) = B, \tag{2}$$

$$u(t_i+) = J_i(u(t_i)), \quad u'(t_i+) = M_i(u'(t_i)), \quad i = 1, \dots, p, \tag{3}$$

where $J_i, M_i \in C^0(\mathbb{R})$ for each $i = 1, \dots, p$ ($C^0(\mathbb{R})$ is the set of real valued functions which are continuous on $\mathbb{R}$) and f has time singularities at the points $t = 0$ and $t = T$. More precisely, we assume that f satisfies the Carathéodory conditions on each set $[a,b] \times \mathbb{R}^2$, where $[a,b] \subset (0,T)$, but f does not satisfy the Carathéodory conditions on $[0,T] \times \mathbb{R}^2$. Recall that f satisfies the Carathéodory conditions on $[a,b] \times \mathbb{R}^2$ if

(i) $f(\cdot, x, y) : [a,b] \to \mathbb{R}$ is measurable for all $(x,y) \in \mathbb{R}^2$;

(ii) $f(t, \cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ is continous for a. e. $t \in [a,b]$;

(iii) for each compact set $K \subset \mathbb{R}^2$ there is a function $m_K \in L[a,b]$ such that $|f(t,x,y)| \le m_K(t)$ for a. e. $t \in [a,b]$ and all $(x,y) \in K$,

(where $L(E)$ denotes the set of all real valued functions which are Lebesgue integrable on the measurable set $E \subset \mathbb{R}$ equipped with the norm

$$\|u\|_1 = \int_E |u(t)| \, \mathrm{d}t \qquad \text{for each } u \in L(E)).$$

We will write $f \in Car([a,b] \times \mathbb{R}^2)$. If for each $[a,b] \subset (0,T)$ we have $f \in Car([a,b] \times \mathbb{R}^2)$, we will write $f \in Car((0,T) \times \mathbb{R}^2)$.

Definition 1 We say that $f \in Car((0,T) \times \mathbb{R}^2)$ has time singularities at the points 0 and T if there exist (x_1, y_1), $(x_2, y_2) \in \mathbb{R}^2$ such that

$$\int_0^\epsilon |f(t, x_1, y_1)| \, \mathrm{d}t = \infty, \qquad \int_{T-\epsilon}^T |f(t, x_2, y_2)| \, \mathrm{d}t = \infty$$

for each sufficiently small $\epsilon > 0$. The points 0 and T are called singular points of f.

For the interval $[0,T]$ and its division D, we define the set PC^0 of all $u : [0,T] \to \mathbb{R}$, where u is continuous on (t_i, t_{i+1}) for $i = 0, \dots, p$, u is continuous from the left at t_i for each $i = 1, \dots, p$ and from the right at 0 and there exists

$u'(t_i+) = \lim_{t \to t_i+} u'(t)$ for each $i = 1, \dots, p$. The set PC^0 is equipped with the supremum norm

$$\|u\|_{PC^0} = \|u\|_\infty = \sup\{|u(t)| : t \in [0, T]\} \qquad \text{for each } u \in PC^0.$$

We consider Banach spaces $X_i = (C^0[t_i, t_{i+1}], \|\cdot\|_\infty)$ for each $i = 0, \dots, p$. There exists an isometric isomorphism $\psi : PC^0 \to \prod_{i=0}^p X_i$ (the space $\prod_{i=0}^p X_i$ is a set of elements of the form $[u_{[0]}, \dots, u_{[p]}]$, where $u_{[i]} \in X_i$ for $i = 0, \dots, p$ and becomes a Banach space with the usual algebraic operations and the norm defined by

$$\|u\| = \max\{\|u_{[i]}\|_\infty : i = 0, \dots, p\} \quad \text{for each } u \in \prod_{i=0}^p X_i).$$

The isomorfism ψ has the form $\psi u = [u_{[0]}, \dots, u_{[p]}]$ for each $u \in PC^0$, where $u_{[i]}$ is defined by

$$u_{[i]} = \begin{cases} u(t) & \text{for} \quad t \in (t_i, t_{i+1}], \\ u(t_i+) & \text{for} \quad t = t_i. \end{cases}$$

Therefore we can consider these spaces as identical and write

$$u = [u_{[0]}, \dots, u_{[p]}] \quad \text{for each } u \in PC^0.$$

For a given set $E \subset \mathbb{R}$, let $C^1(E)$ denote the set of real valued functions defined on E having a continuous first derivative on E, $AC^0(E)$ (or $AC^1(E)$) denotes the set of functions which are absolutely continuous (or have absolutely continuous first derivatives) on E and $AC^0_{\text{loc}}(E)$ (or $AC^1_{\text{loc}}(E)$) denotes the set of functions which are absolutely continuous (or have absolutely continuous first derivatives) on each compact subset of E. Further, we define the space

$$PC^1 = \{u \in PC^0 : u_{[i]} \in C^1[t_i, t_{i+1}] \quad \text{for each } i = 0, \dots, p\}$$

and for $u \in PC^1$ we define u' as a function

$$u' = [(u_{[0]})', \dots, (u_{[p]})'] \in PC^0.$$

Throughout the paper, we will need the following spaces

$$APC^0 = \{u \in PC^0 : u_{[i]} \in AC^0[t_i, t_{i+1}] \text{ for each } i = 0, \dots, p\},$$

$$APC^1 = \{u \in PC^0 : u_{[i]} \in AC^1[t_i, t_{i+1}] \text{ for each } i = 0, \dots, p\},$$

$$APC^0_{\text{loc}} = \{u \in PC^0 : u_{[0]} \in AC^0_{\text{loc}}(t_0, t_1], u_{[p]} \in AC^0_{\text{loc}}[t_p, t_{p+1}), \\ u_{[i]} \in AC^0[t_i, t_{i+1}] \text{ for each } i = 1, \dots, p-1\},$$

$$APC^1_{\text{loc}} = \{u \in PC^0 : u_{[0]} \in AC^1_{\text{loc}}(t_0, t_1], u_{[p]} \in AC^1_{\text{loc}}[t_p, t_{p+1}), \\ u_{[i]} \in AC^1[t_i, t_{i+1}] \text{ for each } i = 1, \dots, p-1\}.$$

For the purpose of proving the existence principle for singular problem we will need this modification of the Arzelà–Ascoli lemma.

Lemma 2 Let us consider the sequence $\{u_n\} \subset PC^0$ satisfying conditions

(i) $\{u_{n,[i]}\}$ is bounded in $C^0[t_i, t_{i+1}]$,

(ii) $\{u_{n,[i]}\}$ is equicontinuous

for each $i = 0, \dots, p$. Then there exists a subsequence $\{u_{k_n}\} \subset \{u_n\}$ and $u \in PC^0$ such that

$$u_{k_n} \to u \quad \text{uniformly on } [0, T].$$

We will search for solutions of the problem (1) – (3) in the space APC^1 which means that each solution has continous first derivatives at the singular points 0 and T.

Definition 3 A function $u \in APC^1$, which satisfies the equation (1) for a. e. $t \in [0, T]$ and fulfils conditions (2) and (3) is called a solution of the problem (1) – (3).

We can find a lot of papers providing the existence of one or more solutions of regular problem (1) – (3), i. e. of problem (1) – (3) with $f \in Car([0, T] \times \mathbb{R}^2)$ or f continuous on $[0, T] \times \mathbb{R}^2$. See [11], [2], [13], [12]. Almost all papers dealing with singular problem (1), (2) or singular impulsive problem (1) – (3) provide the existence of solutions whose first derivatives are not defined at the singular points 0 and T. Here we will call them w–solutions.

Definition 4 A function $u \in APC^1_{\text{loc}}$, which satisfies the equation (1) for a. e. $t \in [0, T]$ and fulfils conditions (2) and (3) is called a w–solution of the problem (1) – (3).

Clearly each solution is a w–solution and each w–solution which moreover belongs to APC^1 is a solution.

As concern existence results for singular impulsive problem (1) – (3), we can only refer to the papers [1], [28], [31] – [33]. The paper [1] establishes existence of positive solutions of impulsive BVPs for both the first order and the second order differential equations. Here f is allowed to have time and space singularities. Other types of conditions which guarantee existence of positive solutions to impulsive Dirichlet problem with space singularities can be found in [28]. In the papers [31] and [32] the second order equation of the generalized Emden–Fowler type has been studied. Under the assumption that the relevant impulsive problem has time singularities the existence of one or more w–solutions has been proved. Existence and multiplicity results for impulsive BVPs in Banach spaces with time singularities are proved in [33].

More results can be found for singular problem (1), (2). The existence of w–solutions is proved e. g. in [3], [8], [18] – [20], [23] – [27]. But we have found just two papers concerning solutions. Habets and Zanolin in [16] have studied the equation

$$u'' = f(t, u), \tag{4}$$

where $f \in C^0((0, T) \times \mathbb{R})$ has time singularities at $t = 0$ and $t = T$. Assuming the existence of lower and upper functions $\sigma_1 \le \sigma_2$ of problem (4), (2), they have obtained the existence of a solution $u \in AC^1[0,T]$ of (4), (2) provided $f(t,x)$ has for $x \in [\sigma_1(t), \sigma_2(t)]$ a majorant which is integrable on $[0,T]$. Yong Zhang in [34] dealt with the equation

$$u'' + p(t)u^\lambda = 0, \tag{5}$$

where $p \in C^0(0,T)$ had time singularities at $t = 0$ and $t = T$, $p(t) \geq 0$ on $(0,T)$ and $\lambda \in (0,1)$. He assumed that

$$0 < \int_0^T t^\lambda (1-t)^\lambda p(t)\, \mathrm{d}t < \infty \tag{6}$$

and proved the existence of a solution $u \in AC^1[0,T]$ of (5), (2).

Let us show the importance of the existence of solutions which are smooth at the singular points 0 and T. It occurs for example if we search for positive, radially symmetric solutions to the nonlinear elliptic partial differential equation

$$\triangle u + g(r,u) = 0 \qquad \text{on} \quad \Omega, \tag{7}$$

where Ω is the open unit disk in $\mathbb{R}^n$ (centered at the origin) and r is the radial distance from the origin. Under the assumption $u'(0) = 0$, radially symmetric solutions of (7) can be found as solutions of the following singular ordinary differential equation

$$u'' + \frac{n-1}{t} u' + g(t,u) = 0 \qquad \text{on} \quad (0,1). \tag{8}$$

We see that just solutions of (8) having continuous first derivatives at the singular point 0 have sense for the associated equation (7). In addition, numerical computations ([4], [5]) lead to smooth solutions of singular Dirichlet problems.

The main task of this paper is to provide conditions which imply the existence of solutions (or w–solutions) both for problem (1), (2) and for impulsive problem (1) – (3).

2 Existence principle for singular problem

If we investigate the solvability of singular problems we often construct approximating regular problems whose solutions converge to a solution (a w–solution) of the original singular problem. The next theorem shows which properties of approximating functions f_n imply the existence of a w–solution or a solution of problem (1) – (3). Let

$$n_0 \in \mathbb{N} \text{ satisfying } \frac{1}{n_0} < t_1 \text{ and } t_p < T - \frac{1}{n_0}. \tag{9}$$

For each $n \in \mathbb{N}$, $n \geq n_0$ we denote

$$\triangle_n = [0, \frac{1}{n}) \cup (T - \frac{1}{n}, T] \tag{10}$$

and we consider a (regular) problem

$$u''(t) + f_n(t, u(t), u'(t)) = 0, \quad (2), \quad (3), \tag{11}$$

where $f_n \in Car([0,T] \times \mathbb{R}^2)$.

Definition 5 A function $u \in APC^1$, which satisfies the differential equation from (11) for a. e. $t \in [0,T]$ and fulfils (2) and (3) is called a solution of the problem (11).

Theorem 6 *Assume that $f \in Car((0,T) \times \mathbb{R}^2)$ has time singularites at $t = 0$ and $t = T$,*

$$\left.\begin{array}{r} f_n(t,x,y) = f(t,x,y) \quad \text{for a. e. } t \in [0,T] \setminus \triangle_n \\ \text{and each } x, y \in \mathbb{R}, \ n \geq n_0, \end{array}\right\} \tag{12}$$

$$\left.\begin{array}{l} \text{there exists a bounded set } \Omega \subset PC^1 \text{ such that regular problem (11)} \\ \text{has a solution } u_n \in \Omega \text{ for each } n \in \mathbb{N}, n \geq n_0. \end{array}\right\} \tag{13}$$

Then

(i) there exists $u \in PC^0$ and a subsequence $\{u_{k_n}\} \subset \{u_n\}$ such that

$$\lim_{k \to \infty} \|u_{n_k} - u\|_{PC^0} = 0, \tag{14}$$

and

$$\lim_{k \to \infty} u'_{n_k}(t) = u'(t) \quad \text{locally uniformly on } (0,T), \tag{15}$$

(ii) $u \in APC^1_{\mathrm{loc}}$ is a w–solution of the problem (1) – (3).

Moreover, assume that

$$\left.\begin{array}{l} \text{there exist } \eta > 0, \ \lambda_1, \lambda_2 \in \{1, -1\}, \ y_1, y_2 \in \mathbb{R} \text{ and } \psi_0 \in L[0,T] \\ \lambda_1 \operatorname{sgn}(u'_n - y_1) f_n(t, u_n(t), u'_n(t)) \geq \psi_0(t) \quad \text{a. e. } t \in (0, \eta), \\ \lambda_2 \operatorname{sgn}(u'_n - y_2) f_n(t, u_n(t), u'_n(t)) \geq \psi_0(t) \quad \text{a. e. } t \in (T - \eta, T). \end{array}\right\} \tag{16}$$

Then $u \in APC^1$ is a solution of the problem (1) – (3).

Proof. Let us consider a sequence $\{u_n\} \subset \Omega$ from (13). The boundedness of Ω in PC^1 implies that $\{u_{n,[i]}\}$ satisfies assumptions of Lemma 2 for each $i = 0, \ldots, p$. We get a function $u \in PC^0$ and a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that (14) is valid. Without any loss of generality we can write $\{u_n\} = \{u_{k_n}\}$. Obviously, (2) and

$$u(t_i+) = \lim_{n \to \infty} u_n(t_i+) = \lim_{n \to \infty} J_i(u_n(t_i)) = J_i(u(t_i)) \quad \text{for each } i = 1, \ldots, p.$$

are satisfied.

Now, we will prove (15). Let $\epsilon > 0$ be such that $\epsilon < t_1$ and $t_p < T - \epsilon$ and a sequence $\{v_n\}$ be defined by

$$v_n(t) = u'_n(t) \quad \text{for each } t \in [\epsilon, T - \epsilon]. \tag{17}$$

To prove the relation (15) we consider the space $PC^0([\epsilon, T-\epsilon])$ which is defined in the same way as the space PC^0 where ϵ and $T-\epsilon$ take place of 0 and T. The notation related to PC^0 remains unaltered for $PC^0([\epsilon, T-\epsilon])$ and will be used in this part of the proof, only.

We will prove that $\{v_{n,[i]}\} \subset PC^0([\epsilon, T-\epsilon])$ satisfy the assumptions of Lemma 2 for $i = 1, \dots, p-1$. Since Ω in PC^1 is bounded it follows that $\{v_n\}$ is bounded, too. Let τ_1, $\tau_2 \in [t_i, t_{i+1}]$, where $i \in \{1, \dots, p-1\}$. In view of (17) and (13) we have

$$\begin{aligned} &|v_{n,[i]}(\tau_2) - v_{n,[i]}(\tau_1)| \\ &\quad = \left| \int_{\tau_1}^{\tau_2} f_n(t, u_n(t), u_n'(t))\, \mathrm{d}t \right| = \left| \int_{\tau_1}^{\tau_2} f(t, u_n(t), u_n'(t))\, \mathrm{d}t \right|. \end{aligned} \tag{18}$$

The properties of the function f imply that there exists $h \in L[t_i, t_{i+1}]$ such that

$$\left| \int_{\tau_1}^{\tau_2} f(t, u_n(t), u_n'(t))\, \mathrm{d}t \right| \le \left| \int_{\tau_1}^{\tau_2} h(t)\, \mathrm{d}t \right|.$$

Therefore $\{v_{n,[i]}\}$ is equicontinous sequence for $i = 1, \dots, p-1$. We will prove the equicontinuity of the sequence $\{v_{n,[0]}\} \subset C^0[\epsilon, t_1]$. Let τ_1, $\tau_2 \in [\epsilon, t_1]$. Then (12) implies (18) for $i = 0$ and for $n \ge n_1 \ge n_0$, where $\frac{1}{n_1} < \epsilon$. Since $f \in Car((0,T) \times \mathbb{R}^2)$, there exists $h \in L[\epsilon, t_1]$ such that

$$|f(t, x, y)| \le h(t) \quad \text{for a.e. } t \in [\epsilon, t_1] \text{ and each } |x|, |y| \le K = K(\Omega).$$

Analogously we can prove the equicontinuity of $\{v_{n,[p]}\}$. Thus $\{v_n\}$ satisfies the assumptions of Lemma 2. Hence, by Lemma 2, (17) and (14), we get the subsequence $\{u_{k_n}\}$ such that (15) is valid and

$$u'_{|[\epsilon, T-\epsilon]} \in PC^0[\epsilon, T-\epsilon].$$

The relation (15) implies

$$u'(t_i+) = \lim_{n_k \to \infty} u'_{n_k}(t_i+) = \lim_{n_k \to \infty} M_i(u'_{n_k}(t_i)) = M_i(u'(t_i)), \quad i = 1, \dots, p.$$

Using (12), (14) and (15) we obtain

$$\lim_{n_k \to \infty} f_{n_k}(t, u_{n_k}(t), u'_{n_k}(t)) = f(t, u(t), u'(t)) \quad \text{for a. e. } t \in [0,T]. \tag{19}$$

Let us choose $\tau_i \in (t_i, t_{i+1})$ for each $i = 0, \dots, p$. Then

$$u'_{n_k}(t) - u'_{n_k}(\tau_i) + \int_{\tau_i}^{t} f_{n_k}(s, u_{n_k}(s), u'_{n_k}(s))\, \mathrm{d}s = 0$$

for each $t \in [t_i, t_{i+1}]$. There is a function $m \in L[\epsilon, T-\epsilon]$ such that

$$|f(t, u(t), u'(t))| \le m(t) \quad \text{and} \quad |f_{n_k}(t, u_{n_k}, u'_{n_k}(t))| \le m(t) \quad \text{for each } k \in \mathbb{N}$$

and for a. e. $t \in [\epsilon, T-\epsilon]$. These facts, the relations (19), (15) and Lebesgue dominated convergence theorem imply

$$u'(t) - u'(\tau_i) + \int_{\tau_i}^{t} f(s, u(s), u'(s))\, \mathrm{d}s = 0$$

for each $t \in [t_i, t_{i+1}] \cap [\epsilon, T-\epsilon]$ for $i = 0, \dots, p$. Since $\epsilon > 0$ is an arbitrary number satisfying $\epsilon < t_1$, $t_p < T - \epsilon$, these equalities and the properties of f imply that $u \in APC^1_{\text{loc}}$ and (1) is satisfied for a. e. $t \in [0, T]$.

Assume (16) holds. To prove $u \in APC^1$, we have to show that

$$f(t, u(t), u'(t)) \in L[0, \eta] \quad \text{and} \quad f(t, u(t), u'(t)) \in L[T-\eta, T],$$

for some $\eta > 0$ such that $\eta < t_1$ and $t_p < T - \eta$. We will prove the first relation, only. Let us denote

$V_1 = \{t \in (0, \eta) : f(t, \cdot, \cdot) \text{ is not continuous } \}$,
$V_2 = \{t \in (0, \eta) : t \text{ is an isolated zero of the function } u' - y_1 \}$,
$V_3 = \{t \in (0, \eta) : u''(t) \text{ does not exist or the equation (1) is not satisfied } \}$.

It is easy to see that $\text{meas}(V) = 0$, where $V = V_1 \cup V_2 \cup V_3$. Let us choose arbitrary $t_0 \in (0, \eta) \setminus V$.
CASE A. Let t_0 is an accumulation point of the set of all zeros of the function $u' - y_1$ on the interval $(0, \eta)$. Then there exists a sequence $\{t_m\} \subset (0, \eta)$ such that $\lim_{m\to\infty} t_m = t_0$ and $u'(t_m) = y_1$. The continuity of u' on the interval $(0, \eta)$ implies that $u'(t_0) = y_1$,

$$\lim_{t_m \to t_0} \frac{u'(t_m) - u'(t_0)}{t_m - t_0} = 0$$

and since $t_0 \notin V_3$ it follows

$$0 = u''(t_0) = -f(t_0, u(t_0), u'(t_0)).$$

The relations (12), (14), (15) and $t_0 \notin V_1$ imply

$$\begin{aligned} &\lim_{k\to\infty} f_{n_k}(t_0, u_{n_k}(t_0), u'_{n_k}(t_0)) \\ &\quad = \lim_{k\to\infty} f(t_0, u_{n_k}(t_0), u'_{n_k}(t_0)) = f(t_0, u(t_0), u'(t_0)) = 0 \end{aligned}$$

and thus

$$\begin{aligned} &\lim_{k\to\infty} \lambda_1 \operatorname{sgn}(u'_{n_k}(t_0) - y_1) f_{n_k}(t_0, u_{n_k}(t_0), u'_{n_k}(t_0)) \\ &\quad = \lambda_1 \operatorname{sgn}(u'(t_0) - y_1) f(t_0, u(t_0), u'(t_0)). \end{aligned} \tag{20}$$

CASE B. Let $u'(t_0) \neq y_1$. Assume that $u'(t_0) > y_1$, i. e. $\operatorname{sgn}(u'(t_0) - y_1) = 1$. Then there exists $\tilde{n} \in \mathbb{N}$ such that for each $n_k \geq \tilde{n}$

$$\operatorname{sgn}(u'_{n_k}(t_0) - y_1) = 1$$

holds. In view of (19) we get (20). For the case $u'(t_0) < y_1$ we proceed similarly. We have proved that (20) is valid for a. e. $t_0 \in (0, \eta)$. Let us put

$$\varphi_{n_k}(t) = \lambda_1 \operatorname{sgn}(u'_{n_k}(t) - y_1) f(t, u_{n_k}(t), u'_{n_k}(t)) + |\psi(t)|$$

and

$$\varphi(t) = \lambda_1 \operatorname{sgn}(u'(t) - y_1) f(t, u(t), u'(t)) + |\psi(t)|$$

for each $n_k \in \mathbb{N}$ and a. e. $t \in (0, \eta)$. Then $\varphi_{n_k} \in L[0, \eta]$ and (16), together with (20) implies

$$\varphi_{n_k}(t) \geq 0, \quad \lim_{k \to \infty} \varphi_{n_k}(t) = \varphi(t) \quad \text{for a. e. } t \in (0, \eta).$$

These facts and $u''_{n_k}(t) = (u'_{n_k}(t) - y_1)'$ imply

$$\begin{aligned}\int_0^\eta \varphi_{n_k}(t)\,\mathrm{d}t &= -\lambda_1 \int_0^\eta \operatorname{sgn}(u'_{n_k}(t) - y_1)(u'_{n_k} - y_1)'\,\mathrm{d}t + \int_0^\eta |\varphi(t)|\,\mathrm{d}t \\ &= -\lambda_1 \int_0^\eta |u'_{n_k}(t) - y_1|'\,\mathrm{d}t + \int_0^\eta |\varphi(t)|\,\mathrm{d}t < C = C(\eta, \psi, \Omega, y_1) \in (0, \infty).\end{aligned}$$

Using Fatou lemma we get $\varphi \in L[0, \eta]$, and thus

$$|f(\cdot, u(\cdot), u'(\cdot))| = |\varphi - |\psi|| \in L[0, \eta].$$

Similarly, we get $f(\cdot, u(\cdot), u'(\cdot)) \in L[T - \eta, T]$. □

3 Regular Dirichlet problem

We bring some results which will be exploited in the investigation of singular problem (1) – (3). Since we will apply Theorem 6 we will study approximating regular problems and prove their solvability. Therefore we will consider a regular equation

$$u''(t) + h(t, u(t), u'(t)) = 0, \tag{21}$$

where $h \in Car([0, T] \times \mathbb{R}^2)$, and prove lower and upper functions method for regular problem (21), (2), (3).

Definition 7 A function $u \in APC^1$, which satisfies the equation (21) for a. e. $t \in [0, T]$ and fulfils conditions (2) and (3) is called a solution of the problem (21), (2), (3).

Definition 8 A function $\sigma_k \in APC^1$ is called a lower (upper) function of the problem (21), (2), (3) provided the conditions

$$[\sigma''_k(t) + h(t, \sigma_k(t), \sigma'_k(t))](-1)^k \leq 0 \quad \text{for a. e. } t \in [0, T], \tag{22}$$

$$(\sigma_k(0) - A)(-1)^k \geq 0, \quad (\sigma_k(T) - B)(-1)^k \geq 0, \tag{23}$$

$$\sigma_k(t_i+) = J_i(\sigma_k(t_i)), \; [\sigma'_k(t_i+) - M_i(\sigma'_k(t_i))](-1)^k \leq 0, \quad i = 1, \ldots, p, \tag{24}$$

where $k = 1$ ($k = 2$), are satisfied.

Lemma 9 *Let us suppose that*

$$\left.\begin{array}{l}\sigma_1,\ \sigma_2 \textit{ are lower and upper functions of the problem } (21),\ (2),\ (3)\\ \textit{and } \sigma_1 \le \sigma_2 \textit{ on } [0,T],\end{array}\right\} \quad (25)$$

$$\left.\begin{array}{l}J_i,\ M_i \in C^0(\mathbb{R}), \quad M_i \textit{ are nondecreasing},\\ J_i(\sigma_1(t_i)) \le J_i(x) \le J_i(\sigma_2(t_i)) \quad \textit{for } \sigma_1(t_i) \le x \le \sigma_2(t_i),\\ \textit{for each } i = 1,\dots,p,\end{array}\right\} \quad (26)$$

there exists $m \in L[0,T]$ *such that*

$$|h(t,x,y)| \le m(t) \quad \textit{for a. e. } t \in [0,T],\ \textit{each } x \in [\sigma_1(t),\sigma_2(t)],\ y \in \mathbb{R}. \quad (27)$$

Then the problem (21), (2), (3) *has a solution* $u \in APC^1$ *such that*

$$\sigma_1 \le u \le \sigma_2 \quad \textit{on } [0,T]. \quad (28)$$

Proof. We define an auxiliary functions

$$g(t,x,y) = \begin{cases} -h(t,\sigma_1(t),y) - \omega_1\Big(t, \frac{\sigma_1(t)-x}{\sigma_1(t)-x+1}\Big) - \frac{\sigma_1(t)-x}{\sigma_1(t)-x+1} & \text{for} \quad x < \sigma_1(t), \\ -h(t,x,y) & \text{for} \quad \sigma_1(t) \le x \le \sigma_2(t), \\ -h(t,\sigma_2(t),y) + \omega_2\Big(t, \frac{x-\sigma_2(t)}{x-\sigma_2(t)+1}\Big) + \frac{x-\sigma_2(t)}{x-\sigma_2(t)+1} & \text{for} \quad \sigma_2(t) < x, \end{cases}$$

where

$$\omega_i(t,\epsilon) = \sup\{|h(t,\sigma_i(t),\sigma_i'(t)) - h(t,\sigma_i(t),y)| : |\sigma_i'(t) - y| \le \epsilon\},$$

for a. e. $t \in [0,T]$, and for $\epsilon \in [0,1]$, $i = 1,2$,

$$\sigma(t,x) = \begin{cases} \sigma_1(t) & \text{for} \quad x < \sigma_1(t), \\ x & \text{for} \quad \sigma_1(t) \le x \le \sigma_2(t), \\ \sigma_2(t) & \text{for} \quad \sigma_2(t) < x, \end{cases}$$

for all $t \in [0,T]$, $x \in \mathbb{R}$,

$$\delta(y) = \begin{cases} y & \text{for} \quad |y| \le c, \\ c \operatorname{sgn} y & \text{for} \quad |y| > c, \end{cases}$$

where

$$c = 2 \max_{i=0,\dots,p}\{(t_{i+1} - t_i)^{-1}\}(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty) + \|h\|_1 + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty + 1.$$

We define an auxiliary problem

$$u'' - g(t,u(t),u'(t)) = 0, \quad (29)$$

$$u(0) = A, \quad u(T) = B, \quad (30)$$

$$\left.\begin{array}{l} u(t_i+) - u(t_i) = J_i(\sigma(t_i,u(t_i)) - \sigma(t_i,u(t_i)), \\ u'(t_i+) - u'(t_i) = M_i(\delta(u'(t_i))) - \delta(u'(t_i)), \end{array}\right\} \text{for } i = 1,\dots,p \quad (31)$$

and an operator $T : PC^1 \to PC^1$ by

$$\begin{aligned}(Tu)(t) = &\int_0^T G(t,s)\tilde{g}(s,u(s),u'(s))\,\mathrm{d}s + A + (B-A)\frac{t}{T}\\ &+\sum_{i=1}^{p}\frac{\partial G}{\partial s}(t,t_i)[J_i(\sigma(t_i,u(t_i))) - \sigma(t_i,u(t_i))]\\ &+\sum_{i=1}^{p} G(t,t_i)[M_i(\delta(u'(t_i))) - \delta(u'(t_i))]\end{aligned}$$

for each $u \in PC^1$ and $t \in [0,T]$. Function $G(t,s) : [0,T]^2 \to \mathbb{R}$ is defined by the formula

$$G(t,s) = \begin{cases} \frac{s(t-T)}{T} & \text{for} \quad 0 \le s < t \le T,\\ \frac{t(s-T)}{T} & \text{for} \quad 0 \le t \le s \le T,\end{cases}$$

It can be proved that $T : PC^1 \to K \subset PC^1$, where K is a bounded, closed ball in PC^1. From the properties of functions G, g, M_i, δ, J_i, σ it follows that T is a compact operator. The Schauder fixed point theorem implies that there exists at least one fixed point $u \in K$ of the operator T, i. e. $Tu = u$. It is possible to prove that u is a solution of the problem (29) – (31),

$$\sigma_1 \le u \le \sigma_2$$

and

$$\|u'\|_\infty \le c.$$

Now, it is easy to see that u is a solution of the problem (21), (2), (3). Similar proof is thoroughly lead in Proposition 8 and Theorem 9 in [29]. □

Now, we will prove a priori estimates which enable to extend the existence result of Lemma 9 to more general right–hand sides h subjected one–sided growth restrictions (see Prop. 11).

Lemma 10 *Let us suppose that there exist a, $b \in (0,t_1)$, $a < b$, y_1, $y_2 \in \mathbb{R}$, $c_0 > 0$, $M_i \in C^0(\mathbb{R})$, M_i is nondecreasing for $i = 1,\dots,p$ and*

$$\left.\begin{array}{l} g \in L[0,T] \text{ be nonnegative and}\\ \omega \in C^0[0,\infty) \text{ positive and } \int_0^\infty \frac{\mathrm{d}s}{\omega(s)} = \infty. \end{array}\right\} \tag{32}$$

Then there exists $\rho_0 > c_0$ such that for each $u \in APC^1$ satisfying conditions

$$|u(t)| \le c_0, \quad \text{for each } t \in [0,T] \tag{33}$$

$$|u'(\xi)| \le c_0 \quad \text{for some } \xi \in [a,b], \tag{34}$$

$$\begin{aligned}&-u''(t)\,\mathrm{sgn}(u'(t) - y_1)\\ &\quad \le \omega(|u'(t) - y_1|)(g(t) + |u'(t) - y_1|) \quad \text{for a. e. } t \in [0,b],\end{aligned} \tag{35}$$

$$\begin{aligned}&-u''(t)\,\mathrm{sgn}(u'(t) - y_2)\\ &\quad \ge -\omega(|u'(t) - y_2|)(g(t) + |u'(t) - y_2|) \quad \text{for a. e. } t \in [a,T],\end{aligned} \tag{36}$$

$$u'(t_i+) = M_i(u'(t_i)) \quad i = 1, \dots, p, \tag{37}$$

the estimation

$$|u'(t)| \le \rho_0 \quad \textit{for every } t \in [0, T]$$

is valid.

Proof. We put

$$v_1'(t) = u'(t) - y_1 \quad t \in [0, \xi].$$

Obviously, from (34) we have

$$|v_1'(\xi)| \le |u'(\xi)| + |y_1| \le c_0 + |y_1| = c_1.$$

By virtue of (32) there exists $\rho_1 > c_1$ such that

$$\int_{c_1}^{\rho_1} \frac{\mathrm{d}s}{\omega(s)} > \|g\|_1 + 2c_0 + T|y_1|. \tag{38}$$

We will prove

$$|v_1'(t)| \le \rho_1 \quad \text{for each } t \in [0, \xi]. \tag{39}$$

Let us consider that (39) is not valid. Then there exists an interval $[\alpha, \beta] \subset [0, \xi]$ such that

$$|v_1'(\alpha)| > \rho_1, \quad |v_1'(\beta)| \le c_1 \quad \text{and} \quad |v_1'(t)| \ne 0 \quad \text{for each } t \in [\alpha, \beta].$$

In view of (35) and the definition of v_1' we get

$$-\frac{v_1''(t)\,\mathrm{sgn}\, v_1'(t)}{\omega(|v_1'(t)|)} \le g(t) + |v_1'(t)| \quad \text{for a. e. } t \in [0, \xi].$$

Integrating this inequality on $[\alpha, \beta]$ and substituing $s = |v_1'(t)|$ we get

$$\begin{aligned} \int_{c_1}^{\rho_1} \frac{\mathrm{d}s}{\omega(s)} &\le \int_{|v_1'(\beta)|}^{|v_1'(\alpha)|} \frac{\mathrm{d}s}{\omega(s)} = -\int_{|v_1'(\alpha)|}^{|v_1'(\beta)|} \frac{\mathrm{d}s}{\omega(s)} \\ &\le \|g\|_1 + \left| \int_\alpha^\beta (u_1'(t) - y_1)\,\mathrm{d}t \right| \le \|g\|_1 + 2c_0 + T|y_1|, \end{aligned}$$

which contradicts (38). We will prove that $v_2'(t)$ defined by

$$v_2'(t) = u'(t) - y_2 \quad \text{for each } t \in [\xi, T]$$

is bounded by a constant independently on u'. The relation (34) implies

$$|v_2'(\xi)| \le |u'(\xi)| + |y_2| \le c_0 + |y_2| = c_{20}.$$

In view of (32) there exists $\rho_{20} > c_{20}$ such that

$$\int_{c_{20}}^{\rho_{20}} \frac{\mathrm{d}s}{\omega(s)} > \|g\|_1 + 2c_0 + T|y_2|. \tag{40}$$

Let us consider that there exists $[\alpha, \beta] \subset [\xi, t_1]$ such that

$$|v_2'(\alpha)| \le c_{20}, \quad |v_2'(\beta)| > \rho_{20} \quad \text{and} \quad |v_2'(t)| \ne 0 \quad \text{for each } t \in [\alpha, \beta].$$

Similarly, we can proceed as for $|v_1'|$ and get the contradiction to (40). It is valid

$$|v_2'(t)| \le \rho_{20} \quad \text{for each } t \in [\xi, t_1].$$

By virtue of (37), property of M_1 and an inequality $|v_2'(t_1)| \le \rho_{20}$ we have

$$|v_2'(t_1+)| \le \max\{|M_1(\rho_{20} + y_2)|, |M_1(-\rho_{20} + y_2)|\} + |y_2| = c_{21}.$$

In view of (32) it follows that there exists $\rho_{21} > c_{21}$ such that

$$\int_{c_{21}}^{\rho_{21}} \frac{\mathrm{d}s}{\omega(s)} > \|g\|_1 + 2c_0 + T|y_2|. \tag{41}$$

Let us suppose that there exists $[\alpha, \beta] \subset (t_1, t_2]$ such that

$$|v_2'(\alpha)| \le c_{21}, \quad |v_2'(\beta)| > \rho_{21} \quad \text{and} \quad |v_2'(t)| \ne 0 \quad \text{for each } t \in [\alpha, \beta].$$

We can proceed as in interval $[0, t_1]$ and get contradiction to (41), again. Thus, $|v_2'(t)| \le \rho_{21}$ for each $t \in (t_1, t_2]$. Similarly, we can proceed on the intervals $(t_i, t_{i+1}]$ for $i = 2, \dots, p$ and get constants $\rho_{22}, \dots, \rho_{2p}$ such that

$$|v_2'(t)| \le \rho_{2i} \quad \text{for each } t \in (t_i, t_{i+1}].$$

We put

$$\rho_0 = \max\{\rho_{20}, \dots, \rho_{2p}, \rho_1\} + |y_1| + |y_2|.$$

□

Proposition 11 *Let the conditions* (25), (26), (32) *hold. Assume that there exist*

$$a, b \in [0, t_1],\ a < b,\ y_1, y_2 \in \mathbb{R} \textit{ such that} \tag{42}$$

$$h(t, x, y + y_1)\,\mathrm{sgn}\, y \le \omega(|y|)(g(t) + |y|)$$
$$\textit{for a. e. } t \in [0, b],\ \textit{each } x \in [\sigma_1(t), \sigma_2(t)],\ y \in \mathbb{R}, \tag{43}$$

$$h(t, x, y + y_2)\,\mathrm{sgn}\, y \ge -\omega(|y|)(g(t) + |y|)$$
$$\textit{for a. e. } t \in [a, T],\ \textit{each } x \in [\sigma_1(t), \sigma_2(t)],\ y \in \mathbb{R}. \tag{44}$$

Then the problem (21), (2), (3) *has a solution* $u \in APC^1$ *satisfying* (28) *and there exists* $\rho_0 > 0$ *(from Lemma 10) such that*

$$|u'(t)| \le \rho_0 \quad \textit{for each } t \in [0, T]. \tag{45}$$

Proof. We put

$$c_0 = 2\frac{1+b-a}{b-a}(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty). \tag{46}$$

For c_0, a, b, y_1, y_2, M_i, g and ω, we get from Lemma 10 the existence of a number $\rho_0 > c_0$ having certain properties. We put

$$r_0 = \rho_0 + \|\sigma_1'\|_\infty + \|\sigma_2'\|_\infty.$$

We define $\chi(y) : \mathbb{R} \to \mathbb{R}$ by

$$\chi(y) = \begin{cases} 1 & \text{for} \quad |y| \le r_0, \\ 2 - \frac{y}{r_0} & \text{for} \quad r_0 < |y| < 2r_0, \\ 0 & \text{for} \quad 2r_0 \le |y|, \end{cases}$$

function $\tilde{f}(t,x,y) = \chi(y)h(t,x,y)$ for a. e. $t \in [0,T]$ and each x, $y \in \mathbb{R}$ and consider a problem

$$u''(t) + \tilde{f}(t,u(t),u'(t)) = 0, \quad (2), \quad (3). \tag{47}$$

We will prove that σ_1 from (25) is a lower function of the problem (47). Obviously, it is valid

$$\sigma_1''(t) \ge -h(t,\sigma_1(t),\sigma_1'(t)) = -\chi(\sigma_1'(t))h(t,\sigma_1(t),\sigma_1'(t)) = -\tilde{f}(t,\sigma_1(t),\sigma_1'(t)).$$

Similar inequality holds for σ_2. It follows from the definition $\tilde{f}$ that there exists $\tilde{h} \in L[0,T]$ such that

$$|\tilde{f}(t,x,y)| \le \tilde{h}(t) \quad \text{for a. e. } t \in [0,T], \text{ each } x \in [\sigma_1(t),\sigma_2(t)], \ y \in \mathbb{R}.$$

By virtue of this fact, (25), (26), we can use Lemma 9 where we put $h = \tilde{f}$ and $m = \tilde{h}$. We get a solution $u \in APC^1$ of the problem (47) satisfying (28). We will check the conditions (33) – (37). It follows from (28) that

$$|u(t)| \le \|\sigma_1\|_\infty + \|\sigma_2\|_\infty \quad \text{for each } t \in [0,T]$$

and from the Mean Value Theorem it follows that there exists $\xi \in [a,b]$

$$u'(\xi) = \frac{u(b)-u(a)}{b-a} \le \frac{2(\|\sigma_1\|_\infty + \|\sigma_2\|_\infty)}{b-a}.$$

The relations (33) and (34) hold for c_0 defined in (46). It follows from (47), definition of $\tilde{f}$ and (43) that

$$\begin{aligned} -u''(t)\operatorname{sgn}(u'(t)-y_1) &= \tilde{f}(t,u(t),u'(t))\operatorname{sgn}(u'(t)-y_1) \\ &= \chi(u'(t))h(t,u(t),u'(t))\operatorname{sgn}(u'(t)-y_1) \\ &\le \chi(u'(t))\omega(|u'(t)-y_1|)(g(t)+|u'(t)-y_1|) \\ &\le \omega(|u'(t)-y_1|)(g(t)+|u'(t)-y_1|) \end{aligned}$$

for a. e. $t \in [0,b]$. We get (35). Similarly, the relation (36) can be obtained from (44). The equalities follow from the definition of a solution of the problem (47). We get (45) and thus

$$0 = u''(t) + \tilde{f}(t,u(t),u'(t)) = u''(t) + h(t,u(t),u'(t))$$

for a. e. $t \in [0,T]$. The function $u \in APC^1$ is a solution of the problem (21), (2), (3). □

4 Main results

Next theorems and corrolaries provide existence results for singular impulsive problem (1) – (3). The notion of lower and upper functions of the singular problem (1) – (3) is needed, here. We will understand them by Definition 8, where f takes place h.

Theorem 12 *Let the conditions* (25), (26), (32), (42),

$$f(t,x,y+y_1)\operatorname{sgn} y \le \omega(|y|)(g(t)+|y|) \quad \text{for a. e. } t \in [0,b], \text{ each } x \in [\sigma_1(t), \sigma_2(t)], \ y \in \mathbb{R}, \tag{48}$$

$$f(t,x,y+y_2)\operatorname{sgn} y \ge -\omega(|y|)(g(t)+|y|) \quad \text{for a. e. } t \in [a,T], \text{ each } x \in [\sigma_1(t), \sigma_2(t)], \ y \in \mathbb{R}, \tag{49}$$

be satisfied. Then there exists a w–solution $u \in APC^1_{\mathrm{loc}}$ of the problem (1) – (3) *such that*

$$\sigma_1 \le u \le \sigma_2 \quad \text{on } [0,T] \quad \text{and} \quad |u'| \le \rho_0 \quad \text{on } (0,T), \tag{50}$$

where ρ_0 is the constant (not depending on u) from Lemma 10. Assume in addition that

$$\left.\begin{array}{ll} \text{there exist } \eta > 0 \ \lambda_1, \lambda_2 \in \{1,-1\}, \ y_1, y_2 \in \mathbb{R} \text{ and } \psi \in L[0,T] & \\ \lambda_1 \operatorname{sgn}(y-y_1) f(t,x,y) \ge \psi(t) & \text{for a. e. } t \in (0,\eta), \\ & \text{each } x \in [\sigma_1(t),\sigma_2(t)], \ |y| \le \rho_0, \\ \lambda_2 \operatorname{sgn}(y-y_2) f(t,x,y) \ge \psi(t) & \text{for a. e. } t \in (T-\eta,T), \\ & \text{each } x \in [\sigma_1(t),\sigma_2(t)], \ |y| \le \rho_0, \end{array}\right\} \tag{51}$$

hold. Then $u \in APC^1$ is a solution of the problem (1) – (3).

Proof. We will use Theorem 6. We will define functions f_n. Let us suppose that (9) is valid and $n \in \mathbb{N}$ such that $n \ge n_0$. The set $\triangle_n$ can be expressed as a disjoint union, we write $\triangle_n = \triangle_{n1} \cup \triangle_{n2}$, where

$$\triangle_{n1} = \{t \in \triangle_n : \sigma_1(t) = \sigma_2(t)\},$$

$$\triangle_{n2} = \{t \in \triangle_n : \sigma_1(t) < \sigma_2(t)\}.$$

We put

$$f_n(t,x,y) = \begin{cases} f(t,x,y) & \text{if } t \notin \triangle_n, \\ -\sigma_1''(t) & \text{if } t \in \triangle_{n1}, \\ -\tilde{f}_n(t,x) & \text{if } t \in \triangle_{n2}, \end{cases} \tag{52}$$

for a. e. $t \in [0,T]$, each $x \in \mathbb{R}$, $y \in \mathbb{R}$, where

$$\tilde{f}_n(t,x) = \begin{cases} \sigma_2''(t) & \text{if } x > \sigma_2(t), \\ \frac{(x-\sigma_1(t))\sigma_2''(t) + (\sigma_2(t)-x)\sigma_1''(t)}{\sigma_2(t)-\sigma_1(t)} & \text{if } \sigma_1(t) \le x \le \sigma_2(t), \\ \sigma_1''(t) & \text{if } x < \sigma_1(t). \end{cases}$$

Then the condition (12) is satisfied. We will prove that σ_1 and σ_2 are lower and upper functions of the problem (11) for $n \geq n_0$. It is sufficient to show, that σ_1 and σ_2 satisfy conditions (22) for $h = f_n$. It is valid

$$\sigma_1''(t) + f_n(t, \sigma_1(t), \sigma_1'(t)) = \sigma_1''(t) + f(t, \sigma_1(t), \sigma_1'(t)) \geq 0$$

for a. e. $t \notin \triangle_n$ and

$$\sigma_1''(t) + f_n(t, \sigma_1(t), \sigma_1'(t)) = \sigma_1''(t) - \tilde{f}_n(t, \sigma_1(t)) = \sigma_1''(t) - \sigma_1''(t) = 0$$

for a. e. $t \in \triangle_{n2}$. If $t \in \triangle_{n1}$, then from (25) and continuity of σ_1', σ_2' at the point t it follows that $\sigma_1'(t) = \sigma_2'(t)$. From (32) and (22) for $h = f$ it follows

$$\sigma_1''(t) \geq -f(t, \sigma_1(t), \sigma_1'(t)) = -f(t, \sigma_2(t), \sigma_2'(t)) \geq \sigma_2''(t)$$

for a. e. $t \in \triangle_{n1}$. From these relations it follows

$$\sigma_1''(t) + f_n(t, \sigma_1(t), \sigma_1'(t)) = \sigma_1''(t) - \sigma_1''(t) = 0$$

and

$$\sigma_2''(t) + f_n(t, \sigma_2(t), \sigma_2'(t)) = \sigma_2''(t) - \sigma_2''(t) = 0$$

for a. e. $t \in \triangle_{n1}$. Thus σ_1, σ_2 is a lower and upper function of the problem (11) for $n \geq n_0$, respectively. Without any loss of generality we can assume that $\omega \geq 1$ on $[0, \infty)$ and $g \geq \max\{|\sigma_1''|, |\sigma_2''|\}$ a. e. on $[0, T]$. Then Proposition 11 (where $h = f_n$) yields the assertion (13) and

$$\sigma_1 \leq u_n \leq \sigma_2 \quad \text{on } [0, T] \quad \text{and} \quad |u_n'| \leq \rho_0 \quad \text{on } (0, T),$$

where $u_n \in APC^1$ is a solution of the problem (11) for $n \geq n_0$. It follows from the properties of the constant ρ_0 that $\{u_n\}$ is contained in some bounded subset $\Omega \subset PC^1$. Then assertions (i) and (ii) of Theorem 6 are valid and u satisfies (50). To prove $u \in APC^1$, it is sufficient to verify the validity of (16). It follows from (52) and (51) with $\psi_0 = \min\{\psi, \sigma_1'', \sigma_2''\}$. □

Corollary 13 *Let the condition* (32) *hold. Assume that there exist* A, $B \in \mathbb{R}$, $a_0, \ldots, a_p$, $b_0, \ldots, b_p \in \mathbb{R}$, $a, b \in [0, t_1]$, $a < b$ *such that*

$$f(t, a_i, 0) \geq 0, \; f(t, b_i, 0) \leq 0 \quad \textit{for a. e. } t \in (t_i, t_{i+1}), \quad i = 0, \ldots, p, \tag{53}$$

$$\left.\begin{array}{l} a_0 \leq A, \; a_p \leq B, \; b_0 \geq A, \; b_p \geq B, \\ a_i = J_i(a_{i-1}), \; b_i = J_i(b_{i-1}), \; M_i(0) = 0, \quad i = 1, \ldots, p, \\ a_i \leq b_i, \quad i = 0, \ldots, p, \end{array}\right\} \tag{54}$$

$$J_i(a_{i-1}) \leq J_i(x) \leq J_i(b_{i-1}) \quad \textit{for each } x \in [a_{i-1}, b_{i-1}], \; i = 1, \ldots, p, \tag{55}$$

$$M_i \textit{ is nondecreasing for } i = 1, \ldots, p, \tag{56}$$

$$f(t, x, y) \operatorname{sgn} y \leq \omega(|y|)(g(t) + |y|) \quad \textit{for a. e. } t \in [0, b], \; x \in [a_0, b_0], \; y \in \mathbb{R}, \tag{57}$$

$$\begin{aligned} &f(t, x, y) \operatorname{sgn} y \geq -\omega(|y|)(g(t) + |y|) \\ &\qquad \textit{for a. e. } t \in [a, T] \cap (t_i, t_{i+1}), \; x \in [a_i, b_i], \; i = 0, \ldots, p, \; y \in \mathbb{R}. \end{aligned} \tag{58}$$

Then there exists a w–solution u of the problem (1) – (3) *such that*

$$a_0 \le u(0) \le b_0 \qquad \text{and} \qquad a_i \le u(t) \le b_i \quad \text{for each } t \in (t_i, t_{i+1}]$$

for each $i = 0, \dots, p$ and there exists ρ_0 (from Lemma 10) such that

$$|u'(t)| \le \rho_0 \qquad \text{for each } t \in (0, T).$$

Moreover if the condition

$$\left.\begin{array}{ll} \text{there exist } \eta > 0,\ \lambda_1, \lambda_2 \in \{1, -1\},\ \text{and } \psi \in L[0,T] \text{ such that} & \\ \lambda_1 \operatorname{sgn} y f(t,x,y) \ge \psi(t) & \text{a. e. } t \in (0, \eta), \\ & \text{each } x \in [a_0, b_0],\ |y| \le \rho_0, \\ \lambda_2 \operatorname{sgn} y f(t,x,y) \ge \psi(t) & \text{a. e. } t \in (T-\eta, T), \\ & \text{each } x \in [a_p, b_p],\ |y| \le \rho_0. \end{array}\right\} \tag{59}$$

holds, then u is a solution of the problem (1) – (3) *and satisfies* (45).

Proof. It suffices to put $\sigma_1(t) = a_0$, $\sigma_2(t) = b_0$ for $t \in [0, t_1]$, $\sigma_1(t) = a_i$, $\sigma_2(t) = b_i$ for $t \in (t_i, t_{i+1}]$, $i = 1, \dots, p$ and use Theorem 12. □

Corollary 14 *Let the condition* (32) *hold and let us assume that there exist A, $B \in \mathbb{R}$, $a_0, \dots, a_p$, $b_0, \dots, b_p$, $c_0, \dots, c_p \in \mathbb{R}$, $a, b \in [0, t_1]$, $a < b$ such that*

$$\left.\begin{array}{l} f(t, a_i + (t-t_i)c_i, c_i) \ge 0,\ f(t, b_i + (t-t_i)c_i, c_i) \le 0 \\ \text{for a. e. } t \in (t_i, t_{i+1})\ i = 0, \dots, p, \\ a_0 \le A,\ a_p + (T - t_p)c_p \le B,\ b_0 \ge A,\ b_p + (T-t_p)c_p \ge B, \\ a_i = J_i(a_{i-1} + (t_i - t_{i-1})c_{i-1}),\ b_i = J_i(b_{i-1} + (t_i - t_{i-1})c_{i-1}), \\ M_i(c_{i-1}) = c_i, \quad i = 1, \dots, p, \\ a_i \le b_i, \quad i = 0, \dots, p \end{array}\right\} \tag{60}$$

$$J_i(a_{i-1} + (t_i - t_{i-1})c_{i-1}) \le J_i(x) \le J_i(b_{i-1} + (t_i - t_{i-1})c_{i-1}) \tag{61}$$

for each $x \in [a_{i-1} + (t_i - t_{i-1})c_{i-1}, b_{i-1} + (t_i - t_{i-1})c_{i-1}]$, $i = 1, \dots, p$,

$$M_i \text{ is nondecreasing for } i = 1, \dots, p, \tag{62}$$

$$f(t,x,y) \operatorname{sgn} y \le \omega(|y|)(g(t) + |y|) \tag{63}$$

for a. e. $t \in [0, b]$, $x \in [a_0 + tc_0, b_0 + tc_0]$, $y \in \mathbb{R}$,

$$f(t,x,y) \operatorname{sgn} y \ge -\omega(|y|)(g(t) + |y|) \tag{64}$$

for a. e. $t \in [a, T] \cap (t_i, t_{i+1})$, $x \in [a_i + (t - t_i)c_i, b_i + (t-t_i)c_i]$, $i = 0, \dots, p$, $y \in \mathbb{R}$. Then there exists a w–solution u of the problem (1) – (3) *such that*

$$a_0 \le u(0) \le b_0 \qquad \text{and} \qquad a_i + (t-t_i)c_i \le u(t) \le b_i + (t - t_i)c_i$$

for each $t \in (t_i, t_{i+1}]$, each $i = 0, \dots, p$ and there exists ρ_0 (from Lemma 10) such that

$$|u'(t)| \le \rho_0 \qquad \text{for each } t \in (0, T).$$

Moreover if the condition

$$\left.\begin{array}{ll} \text{there exist } \eta>0,\ \lambda_1,\lambda_2\in\{1,-1\},\ \text{and } \psi\in L[0,T] \text{ such that} & \\ \lambda_1 \operatorname{sgn} y f(t,x,y) \geq \psi(t) & \text{a. e. } t\in(0,\eta), \\ & \text{each } x\in[a_0+tc_0, b_0+tc_0],\ |y|\leq\rho_0, \\ \lambda_2 \operatorname{sgn} y f(t,x,y) \geq \psi(t) & \text{a. e. } t\in(T-\eta,T), \\ & \text{each } x\in[a_p+(t-t_p)c_p, b_p+(t-t_p)c_p], \\ & |y|\leq\rho_0. \end{array}\right\} \tag{65}$$

holds, then u is a solution of the problem (1) – (3) *and satisfies* (45).

Proof. It suffices to put $\sigma_1(t)=a_0+tc_0$, $\sigma_2(t)=b_0+tc_0$ for $t\in[0,t_1]$, $\sigma_1(t)=a_i+(t-t_i)c_i$, $\sigma_2(t)=b_i+(t-t_i)c_i$ for $t\in(t_i,t_{i+1}]$, $i=1,\dots,p$ and use Theorem 12. □

Corollary 15 *Let the condition* (32) *hold and let us assume that there exist $a,b\in[0,t_1]$, $a<b$ such that*

$$\liminf_{x\to-\infty} f(t,x,0)>0, \quad \limsup_{x\to\infty} f(t,x,0)<0, \tag{66}$$

for a. e. $t\in[0,T]$,

$$\left.\begin{array}{l} J_i,\ M_i\in C^0(\mathbb{R}) \text{ are nondecreasing, } M_i(0)=0, \\ \text{there exists } k>0 \text{ such that } \frac{J_i(x)}{x}\geq 1 \quad \text{for each } |x|\geq k, \end{array}\right\} \tag{67}$$

$i=1,\dots,p$,

$$\text{there exists } \omega\in C^0[0,\infty) \text{ such that } \int_0^\infty \frac{\mathrm{d}s}{\omega(s)},$$

for each $r>0$ there exists $g_r\in L[0,T]$, nonnegative, such that

$$\left.\begin{array}{l} \text{for each } |x|\leq r,\ y\in\mathbb{R} \\ f(t,x,y)\operatorname{sgn} y\leq\omega(|y|)(g_r(t)+|y|) \quad \text{for a. e. } t\in[0,b], \\ f(t,x,y)\operatorname{sgn} y\geq-\omega(|y|)(g_r(t)+|y|) \quad \text{for a. e. } t\in[a,T]. \end{array}\right\} \tag{68}$$

Then there exists $r_0>0$ and $\rho_0>0$ such that the problem (1) – (3) *has a w–solution u satisfying*

$$|u|\leq r_0 \quad \text{on } [0,T] \qquad \text{and} \qquad |u'|\leq\rho_0 \quad \text{on } (0,T).$$

Moreover, if

$$\left.\begin{array}{ll} \text{there exist } \eta>0\ \lambda_1,\lambda_2\in\{1,-1\},\ y_1,y_2\in\mathbb{R} \text{ and } \psi\in L[0,T] & \\ \lambda_1 \operatorname{sgn} y f(t,x,y)\geq\psi(t) & \text{for a. e. } t\in(0,\eta), \\ & \text{each } |x|\leq r_0,\ |y|\leq\rho_0, \\ \lambda_2 \operatorname{sgn} y f(t,x,y)\geq\psi(t) & \text{for a. e. } t\in(T-\eta,T), \\ & \text{each } |x|\leq r_0,\ |y|\leq\rho_0, \end{array}\right\} \tag{69}$$

then u is a solution of the problem (1) – (3) *and satisfies* (45).

Proof. The assumptions (66) imply that there exist $\alpha < A$ and $\beta > B$ such that for each $a_i \leq \alpha$, $b_i \geq \beta$ relations (53) holds. Moreover, by (67), we can find a_i, b_i, for $i = 0, \dots, p$, satisfying (54). For example we choose $b_0 \geq \max\{\beta, k\}$, $b_1 = J_1(b_0) \geq b_0$ and $b_i = J_i(b_{i-1}) \geq b_{i-1}$ for $i = 2, \dots, p$ and similarly $a_0 \leq \min\{\alpha, -k\}$, $a_i = J_i(a_{i-1}) \leq a_{i-1}$ for $i = 1, \dots, p$. Then the assertion follows from Corollary 13. □

Corollary 16 *The assertion of Corollary 15 remains unaltered if the assumption* (67) *is replaced by*

$$\left.\begin{array}{l} J_i,\ M_i \in C^0(\mathbb{R}) \text{ are nondecreasing, } M_i(0) = 0, \\ \text{there exists } k > 0,\ \delta \in (0,1] \text{ and } \gamma \in (0,1] \\ \text{such that } \frac{J_i(x)\operatorname{sgn} x}{|x|^\delta} \geq \gamma > 0 \quad \text{for each } |x| \geq k, \end{array}\right\} \tag{70}$$

for each $i = 1, \dots, p$.

Proof. The assumptions (66) imply that there exist $\alpha < A$ and $\beta > B$ such that for each $a_i \leq \alpha$, $b_i \geq \beta$ relations (53) holds. We choose $c > \max\{\beta, k\}$ and show that there exist $b_0, \dots, b_p$ such that

$$b_i \geq c \qquad \text{and} \qquad b_i = J_i(b_{i-1}) \quad \text{for each } i = 1, \dots, p. \tag{71}$$

We will look for b_0 such that (71) holds. According to (70), the validity of relation

$$b_p = J_p(b_{p-1}) \geq c,$$

can be ensured by inequalities $J_p(b_{p-1}) \geq \gamma b_{p-1}^\delta \geq c$, i. e. $b_{p-1} \geq (c/\gamma)^{\frac{1}{\delta}}$. Similarly the relation

$$b_{p-1} = J_{p-1}(b_{p-2}) \geq \left(\frac{c}{\gamma}\right)^{\frac{1}{\delta}}$$

is valid if

$$b_{p-2} \geq \left(\frac{1}{\gamma}\left(\frac{c}{\gamma}\right)^{\frac{1}{\delta}}\right)^{\frac{1}{\delta}} = \left(\frac{1}{\gamma}\right)^{\frac{1}{\delta}+\frac{1}{\delta^2}} \cdot c^{\frac{1}{\delta^2}}.$$

We can proceed this way and get

$$b_0 \geq \left(\frac{1}{\gamma}\right)^{\frac{1}{\delta}+\frac{1}{\delta^2}+\ldots+\frac{1}{\delta^p}} \cdot c^{\frac{1}{\delta^p}}.$$

This inequality ensures that the relation (71) is valid. Similarly there exist $a_0, \dots, a_p$ such that (53) and (54) hold. Then the assertion follows from Corollary 13. □

Example 17 Let us consider the problem (1) – (3), where

$$f(t,x,y) = y[(T-t)^{-\beta} - t^{-\alpha} + h_1(t)] + cy^2 - h_2(t)(x^{2n-1} - d) + h_3(t)$$

for a. e. $t \in [0,T]$, each $x, y \in \mathbb{R}$, where $\alpha \geq 1$, $\beta \geq 1$, $h_1 \in L[0,T]$, $h_2 \in L[0,T]$, $h_2 \geq \epsilon$, h_3 is an essentially bounded measurable function defined a. e. on $[0,T]$, $\epsilon > 0$, $d \in \mathbb{R}$. Further, J_i are defined as follows

$$J_i(x) = k_i x + \tilde{k}_i, \quad k_i > 1,\ \tilde{k}_i \in \mathbb{R},$$

and M_i a nondecreasing function such that $M_i(0) = 0$ for each $i = 1, \dots, p$. We can check that the conditions of Corollary 15 hold.

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