Superlinear mixed BVP with time and space singularities

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Abstract. Motivated by a problem arising in the theory of shallow membrane caps we investigate the solvability of the singular boundary value problem

$$(p(t)u')' + p(t)f(t,u,p(t)u') = 0, \quad \lim_{t\to 0+} p(t)u'(t) = 0, \quad u(T) = 0,$$

where $[0,T] \subset R$, $p \in C[0,T]$ and $f = f(t,x,y)$ can have time singularities at $t = 0$ and/or $t = T$ and space singularities at $x = 0$ and/or $y = 0$. A superlinear growth of f in its space variables x and y is possible. We present conditions for the existence of solutions positive and decreasing on $[0,T)$.

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1. Introduction.

Let $[0,T] \subset R = (-\infty,\infty)$, $\mathcal{D} \subset R^2$. We deal with the singular mixed boundary value problem

$$(p(t)u')' + p(t)f(t,u,p(t)u') = 0, \tag{1.1}$$

$$\lim_{t\to 0+} p(t)u'(t) = 0, \quad u(T) = 0, \tag{1.2}$$

where $p \in C[0,T]$ and f satisfies the Carathéodory conditions on $(0,T) \times \mathcal{D}$. Here, f can have time singularities at $t = 0$ and/or $t = T$ and space singularities at $x = 0$ and/or $y = 0$. We provide sufficient conditions for the existence of solutions of (1.1), (1.2) which are positive and decreasing on $[0,T)$.

Let $[a,b] \subset R$, $\mathcal{M} \subset R^2$. Recall that a real valued function f satisfies the Carathéodory conditions on the set $[a,b] \times \mathcal{M}$ if

(i) $f(\cdot,x,y) : [a,b] \to R$ is measurable for all $(x,y) \in \mathcal{M}$,

(ii) $f(t,\cdot,\cdot) : \mathcal{M} \to R$ is continuous for a.e. $t \in [a,b]$,

(iii) for each compact set $K \subset \mathcal{M}$ there is a function $m_K \in L_1[0,T]$ such that $|f(t,x,y)| \le m_K(t)$ for a.e. $t \in [a,b]$ and all $(x,y) \in K$.
We write $f \in Car([a,b] \times \mathcal{M})$. By $f \in Car((0,T) \times \mathcal{D})$ we mean that $f \in Car([a,b] \times \mathcal{D})$ for each $[a,b] \subset (0,T)$ and $f \notin Car([0,T] \times \mathcal{D})$.

Definition 1.1. Let $f \in Car((0,T) \times \mathcal{D})$.
We say that f has *a time singularity* at $t = 0$ and/or at $t = T$ if there exists $(x,y) \in \mathcal{D}$ such that

$$\int_0^\varepsilon |f(t,x,y)|dt = \infty \quad \text{and/or} \quad \int_{T-\varepsilon}^T |f(t,x,y)|dt = \infty$$

for each sufficiently small $\varepsilon > 0$. The point $t = 0$ and/or $t = T$ will be called *a singular point of* f.
We say that f has *a space singularity* at $x = 0$ and/or at $y = 0$ if

$$\limsup_{x \to 0+} |f(t,x,y)| = \infty \quad \text{for a.e. } t \in [0,T] \text{ and for some } y \in (-\infty, 0)$$

and/or

$$\limsup_{y \to 0-} |f(t,x,y)| = \infty \quad \text{for a.e. } t \in [0,T] \text{ and for some } x \in (0,\infty).$$

Definition 1.2. By *a solution* of problem (1.1), (1.2) we understand a function $u \in C[0,T] \cap C^1(0,T]$ with $pu' \in AC[0,T]$ satisfying conditions (1.2) and fulfilling

$$(p(t)u'(t))' + p(t)f(t,u(t),p(t)u'(t)) = 0 \quad \text{for a.e. } t \in [0,T]. \tag{1.3}$$

The study of equations with the term $(pu')'$ was motivated by a problem arising in the theory of shallow membrane caps, namely

$$(t^3u')' + \frac{t^3}{8u^2} - a_0\frac{t^3}{u} - b_0t^{2\gamma-1} = 0, \quad \lim_{t \to 0+} t^3u'(t) = 0, \; u(1) = A,$$

where $a_0 \ge 0, b_0 > 0, A > 0, \gamma > 1$.

Singular mixed problem (1.1), (1.2) was studied for example in the works [1, 6] and special cases of (1.1), (1.2) were investigated in [3, 4, 5, 7]. In [2] we can find a mixed problem with ϕ-Laplacian and a real parameter. Here, we generalize the existence results of [7] and extend those of the work [1]. We offer new and rather simple conditions (in comparision with those in [1]), which guarantee the existence of positive solutions of the singular problem (1.1), (1.2) provided both time and space singularities are allowed.

2. Approximating regular problem.

First, we will study the auxiliar regular mixed problem

$$(2.1) \qquad (q(t)u')' + h(t, u, q(t)u') = 0, \quad u'(0) = 0, \ u(T) = 0,$$

where $q \in C[0,T]$ is positive on $[0,T]$ and $h \in Car([0,T] \times R^2)$. In order to prove the solvability of problem (2.1) we will modify the classical lower and upper functions method (see e.e. [5]).

Definition 2.1. A *solution* of the regular problem (2.1) is defined as a function $u \in C^1[0,T]$ with $qu' \in AC[0,T]$ sastisfying $u'(0) = u(T) = 0$ and fulfilling $(q(t)u'(t))' + h(t, u(t), q(t)u'(t)) = 0$ for a.e. $t \in [0,T]$.

Definition 2.2. A function $\sigma \in C[0,T]$ is called *a lower function of* (2.1) if there exists a finite set $\Sigma \subset (0,T)$ such that $q\sigma' \in AC_{loc}([0,T] \setminus \Sigma)$, $\sigma'(\tau+), \sigma'(\tau-) \in R$ for each $\tau \in \Sigma$,

$$(2.2) \qquad (q(t)\sigma'(t))' + h(t, \sigma(t), q(t)\sigma'(t)) \geq 0 \quad \text{for a.e. } t \in [0,T]$$

and

$$(2.3) \qquad \sigma'(0) \geq 0, \quad \sigma(T) \leq 0, \quad \sigma'(\tau-) < \sigma'(\tau+) \quad \text{for each } \tau \in \Sigma.$$

If the inequalities in (2.2) and (2.3) are reversed, then σ is called *an upper function of* (2.1).

Theorem 2.3. (Lower and upper functions method) *Let σ_1 and σ_2 be a lower function and an upper function for problem* (2.1) *such that $\sigma_1 \leq \sigma_2$ on $[0,T]$. Assume also that there is a function $\psi \in L_1[0,T]$ such that*

$$(2.4) \qquad |h(t,x,y| \leq \psi(t) \quad \textit{for a.e. } t \in [0,T], \ \textit{all } x \in [\sigma_1(t), \sigma_2(t)], \ y \in R.$$

Then problem (2.1) *has a solution $u \in C^1[0,T]$ satisfying $qu' \in AC[0,T]$ and*

$$(2.5) \qquad \sigma_1(t) \leq u(t) \leq \sigma_2(t) \quad \textit{for } t \in [0,T].$$

Proof. *Step* 1. For a.e. $t \in [0,T]$ and each $x, y \in R$, $\varepsilon \in [0,1], i = 1,2$, put

$$w_i(t,\varepsilon) = \sup\{|h(t, \sigma_i(t), q(t)\sigma_i'(t)) - h(t, \sigma_i(t), y)| : |q(t)\sigma_i'(t) - y| \leq \varepsilon\},$$

$$h^*(t,x,y) = \begin{cases} h(t, \sigma_2(t), y) - w_2(t, \frac{x-\sigma_2(t)}{x-\sigma_2(t)+1}) - \frac{x-\sigma_2(t)}{x-\sigma_2(t)+1} & \text{for} \quad x > \sigma_2(t) \\ h(t,x,y) & \text{for} \quad \sigma_1(t) \leq x \leq \sigma_2(t) \\ h(t, \sigma_1(t), y) + w_1(t, \frac{\sigma_1(t)-x}{\sigma_1(t)-x+1}) + \frac{\sigma_1(t)-x}{\sigma_1(t)-x+1} & \text{for} \quad x < \sigma_1(t) \end{cases}$$

and consider the auxiliary problem

$$(2.6) \qquad (q(t)u')' + h^*(t, u, q(t)u') = 0, \quad u'(0) = 0, \ u(T) = 0.$$

Define the operator $\mathcal{F} : C^1[0,T] \to C^1[0,T]$ by

$$(\mathcal{F}u)(t) = \int_t^T \frac{1}{q(\tau)} \int_0^\tau h^*(s, u(s), q(s)u'(s))dsd\tau. \tag{2.7}$$

Solving (2.6) is equivalent to finding a fixed point of the operator $\mathcal{F}$. Moreover $h^* \in Car([0,T] \times R^2)$ and there exists $\psi^* \in L_1[0,T]$ such that

$$|h^*(t,x,y)| \le \psi^*(t) \quad \text{for a.e. } t \in [0,T] \text{ and each } x, y \in R.$$

Therefore $\mathcal{F}$ is continuous and compact and the Schauder fixed point theorem yields a fixed point u of $\mathcal{F}$. By (2.7),

$$u(t) = \int_t^T \frac{1}{q(\tau)} \int_0^\tau h^*(s, u(s), q(s)u'(s))dsd\tau \quad \text{for } t \in [0,T],$$

which implies that u is a solution of (2.6).

Step 2. We prove that u satisfies the equation in (2.1). Put $v = u - \sigma_2$ on $[0,T]$ and assume that $\max\{v(t) : t \in [0,T]\} = v(t_0) > 0$. Since $\sigma_2(T) \ge 0$ and $u(T) = 0$, we can assume that $t_0 \in [0,T)$. Hence $v'(t_0) = 0$ and we can find $\delta > 0$ such that for $t \in (t_0, t_0 + \delta)$

$$v(t) > 0, \quad |q(t)v'(t)| < \frac{v(t)}{v(t)+1} < 1.$$

Then for a.e. $t \in (t_0, t_0 + \delta)$ we get

$$(q(t)v'(t))' = -h^*(t, u(t), q(t)u'(t)) - (q(t)\sigma_2'(t))' = -h(t, \sigma_2(t), q(t)u'(t))$$
$$-(q(t)\sigma_2'(t))' + w_2\left(t, \frac{v(t)}{v(t)+1}\right) + \frac{v(t)}{v(t)+1} > 0.$$

Therefore

$$0 < \int_{t_0}^t (q(s)v'(s))'ds = q(t)v'(t)$$

for each $t \in (t_0, t_0 + \delta)$, which contradicts the fact that $v(t_0)$ is the maximal value of v. So $u \le \sigma_2$ on $[0,T]$. The inequality $\sigma_1 \le u$ on $[0,T]$ can be proved analogously. Using the definition of h^* we see that u is also a solution of (2.1). □

3. Main result.

We are interested in positive and decreasing solutions of singular problem (1.1), (1.2) and hence the following existence result will be proved under the assumptions

$$p \in C[0,T], \ p > 0 \text{ on } (0,T], \ \frac{1}{p} \in L_1[0,T], \tag{3.1}$$

and

$$\text{(3.2)} \qquad \begin{cases} \mathcal{D} = (0,\infty) \times (-\infty, 0), \;\; f \in Car((0,T) \times \mathcal{D}), \\ f \text{ can have time singularities at } t = 0, \;\; t = T \\ \text{and space singularities at } x = 0, \;\; y = 0. \end{cases}$$

Theorem 3.1. (Existence result) *Let* (3.1), (3.2) *hold. Assume that there exist* $\varepsilon \in (0,1)$, $\nu \in (0,T)$, $c \in (\nu, \infty)$ *such that*

$$\text{(3.3)} \qquad f(t, P(t), -c) = 0 \quad \textit{for a.e. } t \in [0,T],$$

$$\text{(3.4)} \qquad 0 \le f(t,x,y) \quad \textit{for a.e. } t \in [0,T], \;\; \textit{all } x \in (0, P(t)], \;\; y \in [-c, 0),$$

$$\text{(3.5)} \qquad \varepsilon \le f(t,x,y) \quad \textit{for a.e. } \;\; t \in [0,\nu], \;\; \textit{all } x \in (0, P(t)], \;\; y \in [-\nu, 0),$$

where

$$P(t) = c \int_t^T \frac{ds}{p(s)}.$$

Then problem (1.1), (1.2) *has a positive decreasing solution* $u \in C[0,T]$ *with* $pu' \in AC[0,T]$ *satisfying*

$$\text{(3.6)} \qquad 0 < u(t) \le P(t), \;\; -c \le p(t)u'(t) < 0 \;\; \textit{for } t \in (0,T).$$

Proof. Let $k \in N$, where N is the set of all natural numbers and let $k \ge \frac{3}{T}$.
Step 1. Approximate solutions. For $x, y \in R$ put

$$\alpha_k(x) = \begin{cases} P(t) & \text{if} \quad x > P(t) \\ x & \text{if} \quad \frac{1}{k} \le x \le P(t) \\ \frac{1}{k} & \text{if} \quad x < \frac{1}{k} \end{cases},$$

and

$$\beta_k(y) = \begin{cases} -\frac{1}{k} & \text{if} \quad y > -\frac{1}{k} \\ y & \text{if} \quad -c \le y \le -\frac{1}{k} \\ -c & \text{if} \quad y < -c \end{cases},$$

and

$$\gamma(y) = \begin{cases} \varepsilon & \text{if} \quad y \ge -\nu \\ \varepsilon \frac{c+y}{c-\nu} & \text{if} \quad -c < y < -\nu \\ 0 & \text{if} \quad y \le -c \end{cases}.$$

For a.e. $t \in [0,T]$ and $x, y \in R$ define

$$f_k(t,x,y) = \begin{cases} \gamma(y) & \text{if} \quad t \in [0, \frac{1}{k}) \\ f(t, \alpha_k(x), \beta_k(y)) & \text{if} \quad t \in [\frac{1}{k}, T - \frac{1}{k}] \\ 0 & \text{if} \quad t \in (T - \frac{1}{k}, T] \end{cases}$$

and

$$p_k(t) = \begin{cases} \max\{p(t), p(\frac{1}{k})\} & \text{if} \quad t \in [0, \frac{1}{k}) \\ p(t) & \text{if} \quad t \in [\frac{1}{k}, T] \end{cases}.$$

Then $p_k f_k \in Car([0,T] \times R^2)$ and there is $\psi_k \in L_1[0,T]$ such that

$$|p_k(t) f_k(t,x,y)| \le \psi_k(t) \quad \text{for a.e. } t \in [0,T], \text{ all } x, y \in R. \tag{3.7}$$

We have got a sequence of auxiliary regular problems

$$(p_k(t)u')' + p_k(t) f_k(t, u, p_k(t)u') = 0, \quad u'(0) = 0, \ u(T) = 0, \tag{3.8}$$

for $k \in N$, $k \ge \frac{3}{T}$. Put

$$\sigma_1(t) = 0, \ \sigma_{2k}(t) = c \int_t^T \frac{ds}{p_k(s)} \quad \text{for } t \in [0,T].$$

Then $p_k(t)\sigma'_{2k}(t) = -c$ for $t \in [0,T]$ and conditions (3.3) and (3.4) yield

$$p_k(t) f_k(t, 0, 0) \ge 0, \quad p_k(t) f_k(t, \sigma_{2k}(t), -c) = 0 \quad \text{for a.e. } t \in [0,T].$$

Hence σ_1 and σ_{2k} are lower and upper functions of (3.8). By Theoremm 2.3 problem (3.8) has a solution $u_k \in C^1[0,T]$ satisfying

$$0 \le u_k(t) \le \sigma_{2k}(t) \quad \text{for } t \in [0,T]. \tag{3.9}$$

Note that since $p_k \in C[0,T]$ is positive on $[0,T]$, we have $\sigma_{2k} \in C^1[0,T]$.

Step 2. A priori estimates of approximate solutions. The conditions (3.9) and $u_k(T) = \sigma_{2k}(T) = 0$ give

$$p_k(t) \frac{u_k(T) - u_k(t)}{T - t} \ge p_k(t) \frac{\sigma_{2k}(T) - \sigma_{2k}(t)}{T - t},$$

which yields $p_k(T)u'_k(T) \ge p_k(T)\sigma'_{2k}(T) = -c$. Further, by (3.8), $p_k(0)u'_k(0) = 0$. Since $p_k u'_k$ is nonincreasing on $[0,T]$, we have proved

$$-c \le p_k(t) u'_k(t) \le 0 \quad \text{on } [0,T]. \tag{3.10}$$

Due to $p_k(0)u'_k(0) = 0$, there is $t_k \in (0,T]$ such that

$$-\nu \le p_k(t) u'_k(t) \le 0 \quad \text{for } t \in [0, t_k].$$

If $t_k \geq \nu$, we get by (3.5)

$$p_k(t)u_k'(t) \leq -\varepsilon \int_0^t p(s)ds \quad \text{for } t \in [0,\nu].$$

Assume that $t_k < \nu$ and $p_k(t)u_k'(t) < -\nu$ for $t \in (t_k,\nu]$. Then

$$p_k(t)u_k'(t) \leq -\varepsilon \int_0^t p(s)ds \quad \text{for } t \in [0,t_k]$$

and, since $-\nu < -\varepsilon t$ for $t \in (t_k,\nu]$, we get

$$p_k(t)u_k'(t) \leq -\varepsilon t \text{ for } t \in (t_k,\nu].$$

Choose an arbitrary compact interval $[a,T] \subset (0,T]$ and denote

$$m = \min\{p(t) : t \in [a,T]\}, \quad M = \max\{p(t) : t \in [a,T]\},$$

$$d = \min\{a, \nu, \int_0^a p(s)ds, \int_0^\nu p(s)ds\}.$$

Using the fact that $p_k u_k'$ is nonincreasing on $[0,T]$ we obtain by (3.10) and the above inequalities $-c \leq p_k(t)u_k'(t) \leq -\varepsilon d$ for $t \in [a,T]$ and hence, for each sufficiently large k, we get

$$-\frac{c}{m} \leq u_k'(t) \leq -\frac{\varepsilon d}{M} \quad \text{for } t \in [a,T] \tag{3.11}$$

and

$$(T-t)\frac{\varepsilon d}{M} \leq u_k(t) \leq (T-t)\frac{c}{m} \quad \text{for } t \in [a,T]. \tag{3.12}$$

Step 3. *Convergence of a sequence of approximate solutions.* Consider the sequence $\{u_k\}$. Choose an arbitrary compact interval $J \subset (0,T)$. By virtue of (3.11) and (3.12) there is $k_J \in N$ such that for each $k \in N$, $k \geq k_J$

$$\begin{cases} \frac{1}{k_J} \leq u_k(t) \leq k_J, \quad -k_J \leq u_k'(t) \leq -\frac{1}{k_J}, \\ \qquad -c \leq p_k(t)u_k'(t) \leq -\frac{1}{k_J} \quad \text{for } t \in J, \end{cases} \tag{3.13}$$

and hence there is $\psi \in L_1(J)$ such that

$$|p_k(t)f_k(t,u_k(t),p_k(t)u_k'(t))| \leq \psi(t) \quad \text{a.e. on } J. \tag{3.14}$$

Using conditions (3.13), (3.14) we see that the sequences $\{u_k\}$ and $\{p_k u_k'\}$ are equibounded and equicontinuous on J. Therefore by the Arzelà-Ascoli theorem and the diagonalization principle we can choose $u \in C(0,T)$ and a subsequences of $\{u_k\}$ and of $\{p_k u_k'\}$ which we denote for the simplicity in the same way such that

$$\lim_{k\to\infty} u_k = u, \quad \lim_{k\to\infty} p_k u_k' = pu' \quad \text{locally uniformly on } (0,T). \tag{3.15}$$

Having in mind (3.9), (3.10), (3.13) and the fact that

$$\lim_{k\to\infty} p_k(t) = p(t), \quad \lim_{k\to\infty} \sigma_{2k}(t) = P(t) \quad \text{for } t \in [0,T] \tag{3.16}$$

we get (3.6).

Step 4. *Convergence of a sequence of approximate problems.* Choose an arbitrary $\xi \in (0,T)$ such that

$$f(\xi,\cdot,\cdot) \quad \text{is continuous on } (0,\infty)\times(-\infty,0).$$

There exists a compact interval $J_\xi \subset (0,T)$ with $\xi \in J_\xi$ and, by (3.13), we can find $k_\xi \in N$ such that and for each $k \geq k_\xi$

$$u_k(\xi) \geq \frac{1}{k_\xi}, \quad p_k(\xi)u_k'(\xi) \leq -\frac{1}{k_\xi}, \quad J_\xi \subset [\frac{1}{k}, T-\frac{1}{k}].$$

Therefore

$$f_k(\xi, u_k(\xi), p_k(\xi)u_k'(\xi)) = f(\xi, u_k(\xi), p_k(\xi)u_k'(\xi))$$

and, due to (3.15), (3.16), we have for a.e. $t \in (0,T)$

$$\lim_{k\to\infty} p_k(t) f_k(t, u_k(t), p_k(t)u_k'(t)) = p(t)f(t,u(t),p(t)u'(t)). \tag{3.17}$$

Choose an arbitrary $s \in (0,T)$. Then there exists a compact interval $J_s \subset (0,T)$ containing s and (3.14) holds for $J = J_s$ and for all sufficiently large k. By virtue of (3.8) we get

$$p_k\left(\frac{T}{2}\right)u_k'\left(\frac{T}{2}\right) - p_k(s)u_k'(s) = \int_{\frac{T}{2}}^{s} p_k(\tau)f_k(\tau,u_k(\tau),p_k(\tau)u_k'(\tau))d\tau.$$

Letting $k\to\infty$ and using (3.14)–(3.17) and the Lebesgue convergence theorem on J_s we get for an arbitrary $s \in (0,T)$

$$p\left(\frac{T}{2}\right)u'\left(\frac{T}{2}\right) - p(s)u'(s) = \int_{\frac{T}{2}}^{s} p(\tau)f(\tau,u(\tau),p(\tau)u'(\tau))d\tau. \tag{3.18}$$

Step 5. *Properties of* u *and* pu'. By virtue of (3.18) we have $pu' \in AC_{loc}(0,T)$ and

$$(p(t)u'(t))' + p(t)f(t,u(t),p(t)u'(t)) = 0 \quad \text{for a.e. } t \in (0,T). \tag{3.19}$$

According to (3.8) and (3.10) we have for each $k \geq \frac{3}{T}$

$$\int_0^T p_k(s) f_k(s, u_k(s), p_k(s) u_k'(s)) ds = -p_k(T) u_k'(T) \leq c,$$

which together with (3.4), (3.6) and (3.17) yield, by the Fatou lemma, that $p(t)f(t, u(t), p(t)u'(t)) \in L_1[0,T]$. Therefore, by (3.19), $pu' \in AC[0,T]$. Denote $v = pu'$. Since $v \in C[0,T]$, we have by (3.1) that $u' \in L_1[0,T]$ and consequently $u \in C[0,T] \cap C^1(0,T]$.

Further for each $k \geq \frac{3}{T}$ and $t \in (0,T)$

$$|p_k(t)u_k'(t)| \leq \int_0^t |p_k(s) f_k(s, u_k(s), p_k(s)u_k'(s)) - p(s) f(s, u(s)p(s)u'(s))| ds$$

$$+ \int_0^t |p(s) f(s, u(s), p(s)u'(s))| ds$$

and

$$|u_k(t)| \leq \int_t^T |u_k'(s) - u'(s)| ds + \int_t^T |u'(s)| ds.$$

Hence, by (3.15) and (3.17),

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall t \in (0, \delta) \ \exists k_1 = k_1(\varepsilon, t) \in N :$$

$$|(pu')(t)| \leq |(pu')(t) - (p_{k_1} u_{k_1}')(t)| + |(p_{k_1} u_{k_1}')(t)| < \varepsilon$$

and

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall t \in (T - \delta, T) \ \exists k_2 = k_2(\varepsilon, t) \in N :$$

$$|u(t)| \leq |u(t) - u_{k_2}(t)| + |u_{k_2}(t)| < \varepsilon.$$

This implies

$$u(T) = \lim_{t \to T-} u(t) = 0, \quad (pu')(0) = \lim_{t \to 0+} (pu')(t) = 0. \tag{3.20}$$

□

Remark 3.2. By virtue of (3.20) there is a point $t_0 \in (0,T]$ such that

$$u(t) < P(t), \quad -c < p(t)u'(t) \quad \text{for } t \in [0, t_0).$$

Example. Let $\alpha, \gamma \in (0, \infty)$, $\beta \in [0, \infty)$, $\theta \in (0,1)$. By Theorem 3.1 the problem

$$(t^\theta u')' + t^\theta (u^{-\alpha} + u^\beta + 1)(1 - (-t^\theta u')^\gamma) = 0,$$

$$\lim_{t \to 0+} t^\theta u'(t) = 0, \quad u(1) = 0,$$

has a solution $u \in C[0,1]$ satisfying $t^\theta u' \in AC[0,1]$ and

$$0 < u(t) \le \frac{1 - t^{1-\theta}}{1-\theta}, \quad -1 \le t^\theta u'(t) < 0 \quad \text{for } t \in (0,1).$$

To see this we put $p(t) = t^\theta$, $c = 1$, $\nu = \frac{1}{2}$, $\varepsilon = 1 - (\frac{1}{2})^\gamma$ and $f(t,x,y) = (x^{-\alpha} + x^\beta + 1)(1 - (-y)^\gamma)$.

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