Existence of non-spurious solutions to discrete boundary value problems

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Abstract

This paper investigates discrete boundary value problems (BVPs) involving second-order difference equations and two-point boundary conditions. General theorems guaranteeing the existence and uniqueness of solutions to the discrete BVP are established. The methods involve a sufficient growth condition to yield an *a priori* bound on solutions to a certain family of discrete BVPs. The *a priori* bounds on solutions to the discrete BVP do not depend on the step-size and thus there are no "spurious" solutions. It is shown that solutions of the discrete BVP will converge to solutions of ordinary differential equations.

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1 Introduction

The field of difference equations occupies a central and growing area in modern applicable analysis. The interest in studying difference equations has been created, and is sustained, by two main factors:

1. due to the theory's significant and diverse modelling applications to almost all areas of science, engineering and technology where discrete phenomena abound;

2. from the advent and rise of computers, where differential equations are solved by employing their approximative difference-equation formulations.

Thus the need for, and interest in, scientific advancements in the area is naturally motivated.

This paper investigates the following discrete boundary value problem (BVP) involving second-order difference equations and two-point boundary conditions:

$$\frac{\Delta\nabla y_k}{h^2} = f(t_k, y_k, \frac{\Delta y_k}{h}), \qquad k = 1, \ldots, n-1, \tag{1.1}$$

$$y_0 = A, \quad y_n = B, \tag{1.2}$$

where: f is a continuous, scalar-valued function; the step size is $h = N/n$ with N a positive constant and $n \geq 2$; the grid points are $t_k = kh$ for $k = 0, \ldots, n$; and A, B are given constants in $\mathbb{R}$. The differences are given by:

$$\Delta y_k = \begin{cases} y_{k+1} - y_k, & \text{for } k = 0, \ldots, n-1, \\ 0, & \text{for } k = n; \end{cases}$$

$$\Delta\nabla y_k = \begin{cases} y_{k+1} - 2y_k + y_{k-1}, & \text{for } k = 1, \ldots, n-1, \\ 0, & \text{for } k = 0 \text{ or } k = n. \end{cases}$$

This paper addresses three points of interest regarding the discrete BVP (1.1), (1.2):

- Under what conditions does the discrete BVP (1.1), (1.2) have at least one solution?
- Under what conditions does the discrete BVP (1.1), (1.2) have a unique solution?
- In what sense, if any, will the above solutions to (1.1), (1.2) approximate solutions to the continuous BVP

$$y'' = f(t, y, y'), \qquad t \in [0, N], \tag{1.3}$$

$$y(0) = A, \qquad y(N) = B? \tag{1.4}$$

Particular significance in these points lie in the fact that when a BVP is discretized, strange and interesting changes can occur in the solutions. For example, properties such as existence, uniqueness and multiplicity of solutions may not be shared between the "continuous" differential equation and its related "discrete" difference equation [1, p.520].

A major problem in the numerical approximation of solutions to ordinary differential equations are the existence of "spurious solutions" generated by the approximative difference equation [3, p.417]. These types of solutions do not correspond to any of the solutions to the original differential equation as $h \to 0$. It is desirable to eliminate such irrelevant solutions, if possible.

Sufficiently motivated, the paper is organised as follows.

In Section 2, a general theorem guaranteeing the existence of at least one solution to (1.1), (1.2) is established. The method involves a sufficient growth condition on $|f(t,u,v)|$ in $|u|$ and $|v|$ to yield an *a priori* bound on solutions to a certain family of discrete BVPs. Topological ideas involving homotopy theory and the non-zero property of Brouwer degree are then applied to yield the existence of at least one solution. Next, a theorem is presented that employs a Lipschitz-type condition on f, ensuring that (1.1), (1.2) will have a unique solution.

In Section 3 the *a priori* bound results from Section 2 are applied to show that solutions to the discrete BVP (1.1), (1.2) will converge to solutions of the continuous BVP (1.3), (1.4). The *a priori* bounds on solutions to the discrete BVP do not depend on the step-size and thus there are no spurious solutions. Some examples are presented to illustrate the theory.

For recent and classical results on difference equations and their comparison with differential equations, including existence, uniqueness and spurious solutions, the reader is referred to: [1]-[8], [10]-[16].

A solution to problem (1.3) is a twice continuously differentiable function $y = y(t)$ that satisfies (1.1) for all $t \in [0, N]$.

A solution to problem (1.1) is a vector $\mathbf{y} = (y_0, \ldots, y_n) \in \mathbb{R}^{n+1}$ satisfying (1.1) for $k = 1, \ldots, n-1$.

2 Existence and Uniqueness of Solutions

In this section some new existence and uniqueness results for solutions to (1.1), (1.2).

Our first result involves a sublinear growth condition on $|f(t,u,v)|$ in $|u|$ and $|v|$.

Theorem 2.1 *Let f be continuous on $[0,N] \times \mathbb{R}^2$ and let α, β and K be non-negative constants. If there exist $c, d \in [0,1)$ such that*

$$|f(t,u,v)| \le \alpha |u|^c + \beta |v|^d + K, \quad \forall (t,u,v) \in [0,N] \times \mathbb{R}^2, \tag{2.1}$$

then the discrete BVP (1.1)*,* (1.2) *has at least one solution.*

Proof The BVP (1.1), (1.2) is equivalent to the summation equation

$$y_k = -h \sum_{i=1}^{n-1} G(t_k, s_i) f(s_i, y_i, \frac{\Delta y_i}{h}) + \phi(t_k), \quad k = 0, \ldots, n, \tag{2.2}$$

where $G(t,s)$ is the Green's function for the following discrete BVP

$$\frac{\Delta\nabla y_k}{h^2} = 0, \qquad k = 1, \dots, n-1,$$
$$y_0 = 0, \qquad y_n = 0,$$

and is given explicitly by

$$(2.3) \qquad 0 \le G(t,s) = \frac{1}{N} \begin{cases} t(N-s), & \text{for } 0 \le t \le s \le N, \\ (N-t)s, & \text{for } 0 \le s \le t \le N; \end{cases}$$

and ϕ is the unique solution to the BVP

$$\frac{\Delta\nabla y_k}{h^2} = 0, \qquad k = 1, \dots, n-1,$$
$$y_0 = A, \qquad y_n = B,$$

which is given explicitly by

$$\phi(t_k) = \frac{A}{N}(N - t_k) + \frac{B}{N} t_k, \quad k = 0, \dots, n.$$

Consider the operator $\mathbf{T} : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by

$$T_k(\mathbf{y}) = -h \sum_{i=1}^{n-1} G(t_k, s_i) f(s_i, y_i, \frac{\Delta y_i}{h}) + \phi(t_k), \quad k = 0, \dots, n.$$

Thus we want to show that there exists at least one $\mathbf{y} \in \mathbb{R}^{n+1}$ such that

$$\mathbf{Ty} = \mathbf{y}.$$

To do this, introduce the family of mappings

$$\mathbf{H}_\lambda = I - \lambda \mathbf{T}, \quad \lambda \in [0,1],$$

where I is the identity operator and consider

$$(2.4) \qquad \mathbf{H}_\lambda(\mathbf{y}) = \mathbf{0}, \quad \lambda \in [0,1].$$

We show that $\mathbf{H}_\lambda(\mathbf{y}) \neq \mathbf{0}$ for all $\lambda \in [0,1]$ and all $\mathbf{y} \in \partial B_R$, for some suitable ball $B_R \in \mathbb{R}^{n+1}$. Let us choose a $\lambda \in [0,1]$ and let $\mathbf{y}$ be a solution to the problem (2.4) with this λ. Consider the equivalent summation formulation

$$(2.5) \qquad y_k = -h \sum_{i=1}^{n-1} G(t_k, s_i) \lambda f(s_i, y_i, \frac{\Delta y_i}{h}) + \lambda\phi(t_k), \quad k = 0, \dots, n, \quad \lambda \in [0,1],$$

where G and ϕ are given above. Then, for $k = 0, \ldots, n-1$, we get

$$\begin{aligned}
\left|\frac{\Delta y_k}{h}\right| &= \left|-\sum_{i=1}^{n-1}[\Delta G(t_k, s_i)]\lambda f(s_i, y_i, \frac{\Delta y_i}{h}) + \lambda\frac{\Delta\phi(t_k)}{h}\right| \\
&\le \sum_{i=1}^{n-1}|\Delta G(t_k, s_i)| \left|f(s_i, y_i, \frac{\Delta y_i}{h})\right| + \left|\frac{\Delta\phi(t_k)}{h}\right| \\
&\le \sum_{i=1}^{n-1}|\Delta G(t_k, s_i)| \left[\alpha|y_i|^c + \beta\left|\frac{\Delta y_i}{h}\right|^d + K\right] + \left|\frac{\Delta\phi(t_k)}{h}\right|. \qquad (2.6)
\end{aligned}$$

Put

$$\rho = \max_{k\in\{0,\ldots,n-1\}}\left|\frac{\Delta y_k}{h}\right|, \quad P = \max_{k\in\{0,\ldots,n\}}|\phi(t_k)|.$$

Then $\max_{k\in\{0,\ldots,n\}}|y_k| \le \rho + P$. Further

$$\left|\frac{\Delta\phi(t_k)}{h}\right| = \frac{|B-A|}{N}, \quad \text{for } k = 0, \ldots, n, \qquad (2.7)$$

and

$$\sum_{i=1}^{n-1}|\Delta G(t_k, s_i)| = \frac{h^2}{N}\left(\sum_{i=1}^{k} i + \sum_{i=k+1}^{n-1}(n-i)\right) \le \frac{N}{2}, \quad k = 0, \ldots, n-1. \qquad (2.8)$$

Therefore if we take the maximum in (2.6), we obtain

$$\rho \le \frac{N}{2}\left[\alpha(\rho+P)^c + \beta\rho^d + K\right] + \frac{|B-A|}{N}$$

and so

$$\rho + P \le \frac{N}{2}\left[\alpha(\rho+P)^c + \beta(\rho+P)^d + K\right] + \frac{|B-A|}{N} + P.$$

Hence

$$1 \le \frac{N}{2}\left[(\alpha(\rho+P)^{c-1} + \beta(\rho+P)^{d-1}\right] + \left(\frac{KN}{2} + \frac{|B-A|}{N} + P\right)\rho^{-1} = g(\rho).$$

Since $\lim_{\rho\to\infty} g(\rho) = 0$, there exists $Q > 0$ such that

$$\max_{k\in\{0,\ldots,n-1\}}|y_k| < Q, \quad \max_{k\in\{0,\ldots,n\}}\left|\frac{\Delta y_k}{h}\right| < Q. \qquad (2.9)$$

Define the open ball $\Omega \subset \mathbb{R}^{n+1}$ by

$$\Omega = \left\{\mathbf{y} \in \mathbb{R}^{n+1} : |y_k| < Q, \quad k = 0, \ldots, n, \qquad \left|\frac{\Delta y_k}{h}\right| < Q, \quad k = 0, \ldots, n-1\right\}.$$

The continuity of f implies that $\mathbf{T} : \overline{\Omega} \to \mathbb{R}^{n+1}$ is a continuous map. According to (2.9) we see that for an arbitrary $\lambda \in [0,1]$ there are no solutions to (2.4) (with this λ) belonging to $\partial\Omega$. Hence the following Brouwer degrees are defined and are independent of $\lambda \in [0,1]$ and thus a homotopy principle is applicable [9, Chap.3]. Since $\mathbf{0} \in \Omega$, we get

$$d_B(\mathbf{H}_\lambda, \Omega, \mathbf{0}) = d_B(I - \lambda\mathbf{T}, \Omega, \mathbf{0}) = d_B(\mathbf{H}_0, \Omega, \mathbf{0}) = d(I, \Omega, \mathbf{0}) = 1.$$

Therefore, by the non-zero property of Brouwer degree, there exists at least one solution $\mathbf{y} \in \Omega$ to (2.4) for each $\lambda \in [0,1]$. For $\lambda = 1$ see that (2.4) is equivalent to (1.1), (1.2) and thus the result follows. □

The next theorem allows $|f(t,u,v)|$ to grow linearly in $|u|$ and $|v|$ and thus may apply to certain problems where Theorem 2.1 may be inapplicable.

Theorem 2.2 *Let f be continuous on $[0,N] \times \mathbb{R}^2$ and let α, β and K be non-negative constants. If*

$$|f(t,u,v)| \leq \alpha|u| + \beta|v| + K, \quad \forall (t,u,v) \in [0,N] \times \mathbb{R}^2, \quad \text{and} \tag{2.10}$$

$$\frac{\alpha N^2}{8} + \frac{\beta N}{2} < 1, \tag{2.11}$$

then the discrete BVP (1.1), (1.2) *has at least one solution.*

Proof We argue as in the proof of Theorem 2.1 and derive (2.5). Taking the absolute value in (2.5) and using (2.10) we obtain

$$|y_k| \leq h \sum_{i=1}^{n-1} G(t_k, s_i) \left[\alpha|y_i| + \beta \left| \frac{\Delta y_i}{h} \right| + K \right] + |\phi(t_k)|, \quad k = 0, \ldots, n \tag{2.12}$$

and

$$\left| \frac{\Delta y_k}{h} \right| \leq \sum_{i=1}^{n-1} |\Delta G(t_k, s_i)| \left[\alpha|y_i| + \beta \left| \frac{\Delta y_i}{h} \right| + K \right] + \left| \frac{\Delta\phi(t_k)}{h} \right|. \tag{2.13}$$

Further we have

$$h \sum_{i=1}^{n-1} G(t_k, s_i) = \frac{t_k}{2}(N - t_k) \leq \frac{N^2}{8}, \quad k = 0, \ldots, n, \tag{2.14}$$

and

$$|\phi(t_k)| \leq \max\{|A|, |B|\}, \quad k = 0, \ldots, n. \tag{2.15}$$

Now, by (2.12), (2.14) and (2.15),

$$\max_{k \in \{0,\ldots,n\}} |y_k| \leq \frac{N^2}{8} \left[\alpha \max_{i \in \{1,\ldots,n-1\}} |y_i| + \beta \max_{i \in \{1,\ldots,n-1\}} \left| \frac{\Delta y_i}{h} \right| + K \right] + \max\{|A|, |B|\},$$

and by (2.13), (2.8) and (2.7),

$$\max_{k\in\{0,\ldots,n\}} \left|\frac{\Delta y_k}{h}\right| \le \frac{N}{2}\left[\alpha \max_{i\in\{1,\ldots,n-1\}} |y_i| + \beta \max_{i\in\{1,\ldots,n-1\}} \left|\frac{\Delta y_i}{h}\right| + K\right] + \frac{|B-A|}{N}.$$

Denote

$$\max_{k\in\{0,\ldots,n\}} |y_k| = \rho, \qquad \max_{k\in\{0,\ldots,n-1\}} \left|\frac{\Delta y_k}{h}\right| = \sigma.$$

Then we get

$$\rho \le \frac{N^2}{8}[\alpha\rho + \beta\sigma + K] + \max\{|A|,|B|\}, \qquad \frac{N}{4}\sigma \le \frac{N^2}{8}[\alpha\rho + \beta\sigma + K] + \frac{|B-A|}{4}.$$

Therefore

$$\max\{\rho, \frac{N}{4}\sigma\}\left(1 - \left(\frac{\alpha N^2}{8} + \frac{\beta N}{2}\right)\right) < \frac{KN^2}{8} + \max\{|A|,|B|\} + \frac{|B-A|}{4},$$

which implies by (2.11) that (2.9) holds with

$$Q = \left(1 + \frac{N}{4}\right)\frac{KN^2/8 + \max\{|A|,|B|\} + |B-A|/4}{\alpha N^2/8 + \beta N/2}. \tag{2.16}$$

Now, the rest of the proof follows that of Theorem 2.1.

□

Corollary 2.3 *If f is continuous and bounded on $[0,N]\times\mathbb{R}^2$ then the BVP* (1.1), (1.2) *has at least one solution.*

Proof The result follows from Theorem 2.1 for $c = d = 0$. □

The following theorem gives us conditions for the existence of a unique solution to (1.1), (1.2) and may be considered as a discrete version of [4, Chap. XII, Theorem 4.1], where the uniqueness of solutions to (1.3), (1.4) were established.

Theorem 2.4 *Let f be continuous on $[0,N]\times\mathbb{R}^2$ and let α, β be non-negative constants satisfying* (2.11). *If*

$$|f(t,u,v) - f(t,\tilde{u},\tilde{v})| \le \alpha|u-\tilde{u}| + \beta|v-\tilde{v}|,\ \forall t\in[0,N],\ u,\tilde{u},v,\tilde{v}\in\mathbb{R}, \tag{2.17}$$

then the discrete BVP (1.1), (1.2) *has a unique solution* $\mathbf{y}$ *satisfying* (2.9), *where Q is given by* (2.16) *and* $K = \max_{t\in[0,N]}|f(t,0,0)|$.

Proof See that (2.17) implies that

$$|f(t,u,v)| \le \alpha|u| + \beta|v| + |f(t,0,0)|, \quad \forall (t,u,v)\in[0,N]\times\mathbb{R}^2$$

and thus (2.10) holds. By virtue of (2.11) we have the existence of at least one solution by Theorem 2.2.

Now consider two possible solutions to (1.1), (1.2) given by $\mathbf{y}$ and $\tilde{\mathbf{y}}$ and let $\mathbf{z} = \mathbf{y} - \tilde{\mathbf{y}}$. Now $\mathbf{z}$ must satisfy the BVP

$$\frac{\Delta\nabla z_k}{h^2} = f(t_k, y_k, \frac{\Delta y_k}{h}) - f(t_k, \tilde{y}_k, \frac{\Delta \tilde{y}_k}{h}), \qquad k = 0, \ldots, n-1, \tag{2.18}$$

$$z_0 = 0, \qquad z_n = 0. \tag{2.19}$$

Rearranging (2.18), (2.19) into an equivalent summation equation, taking absolute values and using (2.10) as in the proof of Theorem 2.2 we obtain

$$|z_k| \le h \sum_{i=1}^{n-1} G(t_k, s_i) \ \left[\alpha|z_i| + \beta\left|\frac{\Delta z_i}{h}\right|\right], \quad k = 0, \ldots, n, \tag{2.20}$$

and

$$\left|\frac{\Delta z_k}{h}\right| \le \sum_{i=1}^{n-1} |\Delta G(t_k, s_i)| \ \left[\alpha|z_i| + \beta\left|\frac{\Delta z_i}{h}\right|\right], \quad k = 0, \ldots, n-1. \tag{2.21}$$

Multiplying (2.21) by $N/4$ and using (2.14) in (2.20) and (2.8) in (2.21) we derive

$$\max\left\{|z_k|, \frac{N}{4}\left|\frac{\Delta z_k}{h}\right|\right\}\left(1 - \left(\frac{\alpha N^2}{8} + \frac{\beta N}{2}\right)\right) \le 0$$

and since (2.11) holds we must have $|z_k| = 0$ for $k = 0, \ldots, n$. Thus, the solution is unique. □

Remark 2.5 Note that the conditions in Theorems 2.1, 2.2 and 2.4 do not involve any restrictions on the step-size h (apart from the assumption that $h \le N/2$ which is made so that the problem is well-defined). Thus the conclusions of these theorems apply to those discrete BVPs which do not arise as approximations to continuous BVPs, for example, the case $h = 1$.

Remark 2.6 Note that the the conditions in Theorems 2.1, 2.2 and 2.4 also guarantee the existence and uniqueness of solutions to (1.3), (1.4).

3 Convergence of Solutions

In this section the results of Section 2 are applied to formulate some convergence theorems.

The following result is restated version of [7, Lemma 9.2] (see also [3, pp.414–415]).

Lemma 3.1 *Let n_0 and C be positive constants. Assume that the discrete BVP* (1.1), (1.2) *has a solution $\mathbf{y}^n = (y_0^n, \ldots, y_n^n)$ for $n \ge n_0$ and that the condition*

$$n|\Delta y_k^n| \le C, \quad k = 0, \ldots, n-1, \qquad n \ge n_0 \tag{3.1}$$

is sastisfied. Then there is a subsequence $\{\mathbf{y}^{n_i}\}$ *and a solution* y *to* (1.3), (1.4) *such that*

$$\lim_{i\to\infty} \max_{0\le t\le n_i} |y_k^{n_i} - y(Nt/n_i)| = 0. \tag{3.2}$$

In addition, if it is known that (1.3), (1.4) *has at most one solution, then the original sequence* $\{\mathbf{y}^n\}$ *will converge to* y *in the above sense.*

Proof Choose an arbitrary fixed $n \ge n_0$ and put

$$z_k = y_k^n - \frac{A}{N}(N - t_k) - \frac{B}{N}t_k, \quad k = 0, \dots, n. \tag{3.3}$$

Then $z_0 = z_n = 0$ and by (3.1)

$$n|\Delta z_k| \le C + |B - A| = D, \quad k = 0, \dots, n-1, \qquad n \ge n_0.$$

Therefore

$$|z_k| \le |\Delta z_{k-1}| + |z_{k-1}| \le \frac{D}{n} + \frac{k-1}{n}D = \frac{kD}{n}, \quad \text{for } k = 1, \dots, n-1.$$

We see that

$$|y_k| \le C + |B - A| + \max\{|A|, |B|\}, \quad k = 0, \dots, n, \qquad n \ge n_0.$$

Now, the assertion follows from [7, Lemma 9.2]. □

The following two theorems answer the third question from the Introduction concerning the convergence of solutions for the discrete problem.

Theorem 3.2 *Let the assumptions of Theorem 2.4 hold. Then the discrete problem* (1.1), (1.2) *has a unique solution* $\mathbf{y}^n$ *for each* $n \ge 2$ *and the relevant continuous problem* (1.3), (1.4) *has a unique solution* y *that satisfies*

$$\lim_{n\to\infty} \max_{0\le t\le n} |y_k^n - y(Nt/n)| = 0. \tag{3.4}$$

Proof Since the conditions of Theorem 2.4 hold, the unique solution to (1.1), (1.2) satisfies (2.9) for each $n \ge 2$, which means that the condition (3.1) of Lemma 3.1 is fulfilled. Moreover, by [4, Chap. XII, Theorem 4.1], the continuous problem (1.3), (1.4) has a unique solution because α and β in (2.17) satisfy (2.11). Therefore, by Lemma 3.1, the convergence in (3.4) holds. □

Theorem 3.3 *Let the assumptions of Theorem 2.1 or Theorem 2.2 hold. Then the discrete problem* (1.1), (1.2) *has a solution* $\mathbf{y}^n$ *for each* $n \ge 2$ *and the relevant continuous problem* (1.3), (1.4) *has a solution* y *that satisfies* (3.2).

Proof Since the conditions of Theorem 2.1 or Theorem 2.2 hold, problem (1.1), (1.2) has a solution $\mathbf{y}^n$ satisfying (2.9) for each $n \ge 2$. So, the condition (3.1) of Lemma 3.1 holds and the result follows from there. □

Example 3.4 Consider the discrete BVP

$$\frac{\Delta\nabla y_k}{h^2} = a(t_k)|y_k|^c \operatorname{sign} y_k + b(t_k)\left|\frac{\Delta y_k}{h}\right|^d + g(t_k), \quad k = 0, \ldots, n-1, \tag{3.5}$$

$$y_0 = A, \quad y_n = B, \tag{3.6}$$

where a, b, g are continuous functions on $[0, N]$ and $c, d \in [0, 1)$.

Then, by Theorem 2.1, problem (3.5), (3.6) has at least one solution $\mathbf{y}^n$ for each $n \geq 2$. By Theorem 3.3 there is a solution y to the relevant continuous problem

$$\begin{aligned} y'' &= a(t)|y|^c \operatorname{sign} y + b(t)|y'|^d + g(t), \quad t \in [0, N], \\ & y(0) = A, \quad y(N) = B, \end{aligned}$$

such that (3.2) holds for some sequence $\{\mathbf{y}^{n_i}\}$ of solutions of (3.5), (3.6). □

Example 3.5 Consider the discrete equation

$$\frac{\Delta\nabla y_k}{h^2} = a(t_k)y_k + b(t_k)\frac{\Delta y_k}{h} + g(t_k), \quad k = 0, \ldots, n-1, \tag{3.7}$$

$$y_0 = A, \quad y_n = B, \tag{3.8}$$

where a, b, g are continuous functions on $[0, 1]$. If

$$\max_{t\in[0,1]} |a(t)| + 4 \max_{t\in[0,1]} |b(t)| < 8,$$

then by Theorem 2.4 problem (3.7), (3.8) has a unique solution $\mathbf{y}^n$ for each $n \geq 2$. In addition there is a unique solution y to the relevant continuous problem

$$y'' = a(t)y + b(t)y' + g(t), \quad y(0) = A, \; y(1) = B,$$

and by Theorem 3.2 the convergence in (3.4) holds.

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