Hledání optimálního stromu

Vstup : $2N+1$ nezáporných vah $(p_1, \dots, p_N; q_0, \dots, q_N)$

Výstup: binární vyhledávací stromy $t(i,j)$, které mají minimální cenu vyhledávání pro váhy $(p_{i+1}, \dots, p_j; q_i, \dots, q_j)$

Inicializace;

Pro $0 \le i \le N$: $c[i,i] \leftarrow 0$;
$w[i,i] \leftarrow q_i$
$w[i,j] \leftarrow w[i,j-1] + p_j + q_j$ pro $j = i+1, \dots, N$;

Pro $1 \le j \le N$: $c[j-1,j] \leftarrow w[j-1,j]$;
$r[j-1,j] \leftarrow j$;

for $d \leftarrow 2$ **to** N **do**
  for $j \leftarrow d$ **to** N **do**
    $i \leftarrow j - d$;
    $c[i,j] \leftarrow w[i,j] + \min_{r[i,j-1] \le k \le r[i+1,j]} (c[i,k-1] + c[k,j])$;
    $r[i,j] \leftarrow k$ pro které nastává to minimum;

Vstup: $p_i = (4, 1, 2, 2, 6)$; $q_i = (1, 1, 1, 3, 2, 1)$.
Inicializace:

$$\overset{W}{\begin{bmatrix} 1 & 6 & 8 & 13 & 17 & 24 \\ & 1 & 3 & 8 & 12 & 19 \\ & & 1 & 6 & 10 & 17 \\ & & & 3 & 7 & 14 \\ & & & & 2 & 9 \\ & & & & & 1 \end{bmatrix}} \overset{C}{\begin{bmatrix} & 6 & & & & \\ & & 3 & & & \\ & & & 6 & & \\ & & & & 7 & \\ & & & & & 9 \\ & & & & & \end{bmatrix}} \overset{R}{\begin{bmatrix} & 1 & & & & \\ & & 2 & & & \\ & & & 3 & & \\ & & & & 4 & \\ & & & & & 5 \\ & & & & & \end{bmatrix}}$$

$$d = \quad 2 \quad 3 \quad 4 \quad 5$$
$$j = \quad 2 \quad 3 \quad 4 \quad 5$$
$$i = 2 - 2 = 0$$
$$r[i, j-1] = r[0,1] = 1$$
$$r[i+1, j] = r[1,2] = 2$$

$$W = \begin{bmatrix} 1 & 6 & 8 & 13 & 17 & 24 \\ & 1 & 3 & 8 & 12 & 19 \\ & & 1 & 6 & 10 & 17 \\ & & & 3 & 7 & 14 \\ & & & & 2 & 9 \\ & & & & & 1 \end{bmatrix}$$

$$k = 1 : c[i, k-1] + c[k, j] =$$
$$= c[0,0] + c[1,2] = 3$$

$$C = \begin{bmatrix} 6 & 11 & & & \\ & 3 & & & \\ & & 6 & & \\ & & & 7 & \\ & & & & 9 \end{bmatrix}$$

$$k = 2 : c[i, k-1] + c[k, j] =$$
$$= c[0,1] + c[2,2] = 6$$

$$c[i,j] = c[0,2] \leftarrow w[0,2] + 3 = 11$$
$$r[i,j] = r[0,2] \leftarrow 1$$

$$R = \begin{bmatrix} 1 & 1 & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 5 \end{bmatrix}$$

$d = \quad {\color{blue}2} \quad 3 \quad 4 \quad 5$

$j = \quad 2 \quad {\color{blue}3} \quad 4 \quad 5$

$i = 3 - 2 = 1$

$r[i, j-1] = r[1,2] = 2$

$r[i+1, j] = r[2,3] = 3$

$$W = \begin{bmatrix} 1 & 6 & 8 & 13 & 17 & 24 \\ & 1 & 3 & 8 & 12 & 19 \\ & & 1 & 6 & 10 & 17 \\ & & & 3 & 7 & 14 \\ & & & & 2 & 9 \\ & & & & & 1 \end{bmatrix}$$

$k = 2 : c[i, k-1] + c[k, j] =$

$\qquad = c[1,1] + c[2,3] = 6$

$k = 3 : c[i, k-1] + c[k, j] =$

$\qquad = c[1,2] + c[3,3] = 3$

$$C = \begin{bmatrix} & 6 & 11 & & & \\ & & 3 & {\color{red}11} & & \\ & & & 6 & & \\ & & & & 7 & \\ & & & & & 9 \\ & & & & & \end{bmatrix}$$

$c[i,j] = c[1,3] \leftarrow w[1,3] + 3 = 11$

$r[i,j] = r[1,3] \leftarrow 3$

$$R = \begin{bmatrix} & 1 & 1 & & & \\ & & 2 & {\color{red}3} & & \\ & & & 3 & & \\ & & & & 4 & \\ & & & & & 5 \end{bmatrix}$$

$d = \quad 2 \quad 3 \quad 4 \quad 5$

$j = \quad 2 \quad 3 \quad 4 \quad 5$

$i = 4 - 2 = 2$

$r[i, j-1] = r[2,3] = 3$

$r[i+1, j] = r[3,4] = 4$

$$W = \begin{bmatrix} 1 & 6 & 8 & 13 & 17 & 24 \\ & 1 & 3 & 8 & 12 & 19 \\ & & 1 & 6 & 10 & 17 \\ & & & 3 & 7 & 14 \\ & & & & 2 & 9 \\ & & & & & 1 \end{bmatrix}$$

$$k = 3 : c[i, k-1] + c[k, j] =$$
$$= c[2,2] + c[3,4] = 7$$

$$k = 4 : c[i, k-1] + c[k, j] =$$
$$= c[2,3] + c[4,4] = 6$$

$$C = \begin{bmatrix} & 6 & 11 & & & \\ & & 3 & 11 & & \\ & & & 6 & 16 & \\ & & & & 7 & \\ & & & & & 9 \\ & & & & & \end{bmatrix}$$

$c[i, j] = c[2, 4] \leftarrow w[2, 4] + 6 =$
16

$r[i, j] = r[2, 4] \leftarrow 4$

$$R = \begin{bmatrix} 1 & 1 & & & \\ & 2 & 3 & & \\ & & 3 & 4 & \\ & & & 4 & \\ & & & & 5 \end{bmatrix}$$

$$
\begin{aligned}
d &= \quad 2 \quad 3 \quad 4 \quad 5\\
j &= \quad 2 \quad 3 \quad 4 \quad 5\\
i &= 5 - 2 = 3\\
r[i, j-1] &= r[3,4] = 4\\
r[i+1, j] &= r[4,5] = 5
\end{aligned}
$$

$$
W = \begin{bmatrix}
1 & 6 & 8 & 13 & 17 & 24\\
 & 1 & 3 & 8 & 12 & 19\\
 & & 1 & 6 & 10 & 17\\
 & & & 3 & 7 & 14\\
 & & & & 2 & 9\\
 & & & & & 1
\end{bmatrix}
$$

$$
\begin{aligned}
k = 4 : c[i, k-1] + c[k, j] &=\\
&= c[3,3] + c[4,5] = 9
\end{aligned}
$$

$$
\begin{aligned}
k = 5 : c[i, k-1] + c[k, j] &=\\
&= c[3,4] + c[5,5] = 7
\end{aligned}
$$

$$
C = \begin{bmatrix}
 & 6 & 11 & & & \\
 & & 3 & 11 & & \\
 & & & 6 & 16 & \\
 & & & & 7 & 21\\
 & & & & & 9\\
 & & & & &
\end{bmatrix}
$$

$c[i, j] = c[3,5] \leftarrow w[3,5] + 7 =$
21
$r[i, j] = r[3,5] \leftarrow 5$

$$
R = \begin{bmatrix}
1 & 1 & & & \\
 & 2 & 3 & & \\
 & & 3 & 4 & \\
 & & & 4 & 5\\
 & & & & 5
\end{bmatrix}
$$

$$d = \quad 2 \quad 3 \quad 4 \quad 5$$
$$j = \quad 3 \quad 4 \quad 5$$
$$i = 3 - 3 = 0$$
$$r[i, j-1] = r[0,2] = 1$$
$$r[i+1, j] = r[1,3] = 3$$

$$W = \begin{bmatrix} 1 & 6 & 8 & 13 & 17 & 24 \\ & 1 & 3 & 8 & 12 & 19 \\ & & 1 & 6 & 10 & 17 \\ & & & 3 & 7 & 14 \\ & & & & 2 & 9 \\ & & & & & 1 \end{bmatrix}$$

$$k = 1 : c[i, k-1] + c[k, j] =$$
$$= c[0,0] + c[1,3] = 11$$

$$C = \begin{bmatrix} 6 & 11 & 24 & & \\ & 3 & 11 & & \\ & & 6 & 16 & \\ & & & 7 & 21 \\ & & & & 9 \end{bmatrix}$$

$$k = 2 : c[i, k-1] + c[k, j] =$$
$$= c[0,1] + c[2,3] = 12$$

$$k = 3 : c[i, k-1] + c[k, j] =$$
$$= c[0,2] + c[3,3] = 11$$

$$R = \begin{bmatrix} 1 & 1 & 1 & & \\ & 2 & 3 & & \\ & & 3 & 4 & \\ & & & 4 & 5 \\ & & & & 5 \end{bmatrix}$$

$$d = \quad 2 \quad 3 \quad 4 \quad 5$$
$$j = \quad 3 \quad 4 \quad 5$$
$$i = 4 - 3 = 1$$
$$r[i, j-1] = r[1,3] = 3$$
$$r[i+1, j] = r[2,4] = 4$$

$$W = \begin{bmatrix} 1 & 6 & 8 & 13 & 17 & 24 \\ & 1 & 3 & 8 & 12 & 19 \\ & & 1 & 6 & 10 & 17 \\ & & & 3 & 7 & 14 \\ & & & & 2 & 9 \\ & & & & & 1 \end{bmatrix}$$

$$k = 3 : c[i, k-1] + c[k, j] =$$
$$= c[1,2] + c[3,4] = 10$$

$$k = 4 : c[i, k-1] + c[k, j] =$$
$$= c[1,3] + c[4,4] = 11$$

$$C = \begin{bmatrix} & 6 & 11 & 24 & & \\ & & 3 & 11 & 22 & \\ & & & 6 & 16 & \\ & & & & 7 & 21 \\ & & & & & 9 \\ & & & & & \end{bmatrix}$$

$$c[i, j] = c[1,4] \leftarrow$$
$$w[1,4] + 10 = 22$$
$$r[i, j] = r[1,4] \leftarrow 3$$

$$R = \begin{bmatrix} 1 & 1 & 1 & & \\ & 2 & 3 & 3 & \\ & & 3 & 4 & \\ & & & 4 & 5 \\ & & & & 5 \end{bmatrix}$$

$d = \quad 2 \quad 3 \quad 4 \quad 5$
$j = \quad 3 \quad 4 \quad 5$
$i = 5 - 3 = 2$
$r[i, j-1] = r[2,4] = 4$
$r[i+1, j] = r[3,5] = 5$

$$W = \begin{bmatrix} 1 & 6 & 8 & 13 & 17 & 24 \\ & 1 & 3 & 8 & 12 & 19 \\ & & 1 & 6 & 10 & 17 \\ & & & 3 & 7 & 14 \\ & & & & 2 & 9 \\ & & & & & 1 \end{bmatrix}$$

$k = 4 : c[i, k-1] + c[k, j] =$
$= c[2,3] + c[4,5] = 15$

$$C = \begin{bmatrix} & 6 & 11 & 24 & & \\ & & 3 & 11 & 22 & \\ & & & 6 & 16 & 32 \\ & & & & 7 & 21 \\ & & & & & 9 \\ & & & & & \end{bmatrix}$$

$k = 5 : c[i, k-1] + c[k, j] =$
$= c[2,4] + c[5,5] = 16$

$c[i,j] = c[2,5] \leftarrow$
$w[2,5] + 15 = 32$
$r[i,j] = r[2,5] \leftarrow 4$

$$R = \begin{bmatrix} 1 & 1 & 1 & & \\ & 2 & 3 & 3 & \\ & & 3 & 4 & 4 \\ & & & 4 & 5 \\ & & & & 5 \end{bmatrix}$$

$d = \quad 2 \quad 3 \quad 4 \quad 5$

$j = \quad 4 \quad 5$

$i = 4 - 4 = 0$

$r[i, j-1] = r[0,3] = 1$

$r[i+1, j] = r[1,4] = 3$

$$W = \begin{bmatrix} 1 & 6 & 8 & 13 & 17 & 24 \\ & 1 & 3 & 8 & 12 & 19 \\ & & 1 & 6 & 10 & 17 \\ & & & 3 & 7 & 14 \\ & & & & 2 & 9 \\ & & & & & 1 \end{bmatrix}$$

$$k = 1 : c[i, k-1] + c[k, j] =$$
$$= c[0,0] + c[1,4] = 22$$

$$C = \begin{bmatrix} 6 & 11 & 24 & 35 & \\ & 3 & 11 & 22 & \\ & & 6 & 16 & 32 \\ & & & 7 & 21 \\ & & & & 9 \\ & & & & \end{bmatrix}$$

$$k = 2 : c[i, k-1] + c[k, j] =$$
$$= c[0,1] + c[2,4] = 22$$

$$k = 3 : c[i, k-1] + c[k, j] =$$
$$= c[0,2] + c[3,4] = 18$$

$$R = \begin{bmatrix} 1 & 1 & 1 & 3 & \\ & 2 & 3 & 3 & \\ & & 3 & 4 & 4 \\ & & & 4 & 5 \\ & & & & 5 \end{bmatrix}$$

$d = \quad 2 \quad 3 \quad 4 \quad 5$

$j = \quad 4 \quad 5$

$i = 5 - 4 = 1$

$r[i, j-1] = r[1,4] = 3$

$r[i+1, j] = r[2,5] = 4$

$k = 3 : c[i, k-1] + c[k, j] =$

$= c[1,2] + c[3,5] = 24$

$k = 4 : c[i, k-1] + c[k, j] =$

$= c[1,3] + c[4,5] = 20$

$c[i,j] = c[1,5] \leftarrow$

$w[1,5] + 20 = 39$

$r[i,j] = r[1,5] \leftarrow 4$

$$W = \begin{bmatrix} 1 & 6 & 8 & 13 & 17 & 24 \\ & 1 & 3 & 8 & 12 & 19 \\ & & 1 & 6 & 10 & 17 \\ & & & 3 & 7 & 14 \\ & & & & 2 & 9 \\ & & & & & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 11 & 24 & 35 & \\ & 3 & 11 & 22 & 39 \\ & & 6 & 16 & 32 \\ & & & 7 & 21 \\ & & & & 9 \\ & & & & \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 1 & 1 & 3 & \\ & 2 & 3 & 3 & 4 \\ & & 3 & 4 & 4 \\ & & & 4 & 5 \\ & & & & 5 \end{bmatrix}$$

$d = \quad 2 \quad 3 \quad 4 \quad 5$

$j = \quad 5$

$i = 5 - 5 = 0$

$r[i, j-1] = r[0,4] = 3$

$r[i+1, j] = r[1,5] = 4$

$$W = \begin{bmatrix} 1 & 6 & 8 & 13 & 17 & 24 \\ & 1 & 3 & 8 & 12 & 19 \\ & & 1 & 6 & 10 & 17 \\ & & & 3 & 7 & 14 \\ & & & & 2 & 9 \\ & & & & & 1 \end{bmatrix}$$

$$k = 3 : c[i, k-1] + c[k, j] =$$
$$= c[0,2] + c[3,5] = 32$$

$$C = \begin{bmatrix} & 6 & 11 & 24 & 35 & 56 \\ & & 3 & 11 & 22 & 39 \\ & & & 6 & 16 & 32 \\ & & & & 7 & 21 \\ & & & & & 9 \\ & & & & & \end{bmatrix}$$

$$k = 4 : c[i, k-1] + c[k, j] =$$
$$= c[0,3] + c[4,5] = 33$$

$c[i,j] = c[0,5] \leftarrow$

$w[0,5] + 32 = 56$

$r[i,j] = r[0,5] \leftarrow 3$

$$R = \begin{bmatrix} 1 & 1 & 1 & 3 & 3 \\ & 2 & 3 & 3 & 4 \\ & & 3 & 4 & 4 \\ & & & 4 & 5 \\ & & & & 5 \end{bmatrix}$$

$p = (1, 5, 2, 2, 1)$; $q = (3, 1, 3, 1, 2, 1)$
Inicializace:

$$\begin{array}{ccc} W & C & R \\ \begin{bmatrix} 3 & 5 & 13 & 16 & 20 & 22 \\ & 1 & 9 & 12 & 16 & 18 \\ & & 3 & 6 & 10 & 12 \\ & & & 1 & 5 & 7 \\ & & & & 2 & 4 \\ & & & & & 1 \end{bmatrix} & \begin{bmatrix} 5 & & & & \\ & 9 & & & \\ & & 6 & & \\ & & & 5 & \\ & & & & 4 \\ & & & & \end{bmatrix} & \begin{bmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 4 & \\ & & & & 5 \\ & & & & \end{bmatrix} \end{array}$$

$d = \quad 2 \quad 3 \quad 4 \quad 5$

$j = \quad 2 \quad 3 \quad 4 \quad 5$

$i = 2 - 2 = 0$

$r[i, j-1] = r[0,1] = 1$

$r[i+1, j] = r[1,2] = 2$

$$W = \begin{bmatrix} 3 & 5 & 13 & 16 & 20 & 22 \\ & 1 & 9 & 12 & 16 & 18 \\ & & 3 & 6 & 10 & 12 \\ & & & 1 & 5 & 7 \\ & & & & 2 & 4 \\ & & & & & 1 \end{bmatrix}$$

$k = 1 : c[i, k-1] + c[k, j] =$

$= c[0,0] + c[1,2] = 9$

$k = 2 : c[i, k-1] + c[k, j] =$

$= c[0,1] + c[2,2] = 5$

$$C = \begin{bmatrix} & 5 & 18 & & & \\ & & 9 & & & \\ & & & 6 & & \\ & & & & 5 & \\ & & & & & 4 \\ & & & & & \end{bmatrix}$$

$c[i,j] = c[0,2] \leftarrow w[0,2] + 5 = 18$

$r[i,j] = r[0,2] \leftarrow 2$

$$R = \begin{bmatrix} & 1 & 2 & & & \\ & & 2 & & & \\ & & & 3 & & \\ & & & & 4 & \\ & & & & & 5 \end{bmatrix}$$

$d = \quad 2 \quad 3 \quad 4 \quad 5$

$j = \quad 2 \quad 3 \quad 4 \quad 5$

$i = 3 - 2 = 1$

$r[i, j-1] = r[1,2] = 2$

$r[i+1, j] = r[2,3] = 3$

$$W = \begin{bmatrix} 3 & 5 & 13 & 16 & 20 & 22 \\ & 1 & 9 & 12 & 16 & 18 \\ & & 3 & 6 & 10 & 12 \\ & & & 1 & 5 & 7 \\ & & & & 2 & 4 \\ & & & & & 1 \end{bmatrix}$$

$k = 2 : c[i, k-1] + c[k, j] =$

$= c[1,1] + c[2,3] = 6$

$$C = \begin{bmatrix} & 5 & 18 & & & \\ & & 9 & 18 & & \\ & & & 6 & & \\ & & & & 5 & \\ & & & & & 4 \\ & & & & & \end{bmatrix}$$

$k = 3 : c[i, k-1] + c[k, j] =$

$= c[1,2] + c[3,3] = 9$

$c[i, j] = c[1,3] \leftarrow w[1,3] + 6 =$
18

$r[i, j] = r[1,3] \leftarrow 2$

$$R = \begin{bmatrix} & 1 & 2 & & & \\ & & 2 & 2 & & \\ & & & 3 & & \\ & & & & 4 & \\ & & & & & 5 \end{bmatrix}$$

$d = \quad {\color{blue}2} \quad 3 \quad 4 \quad 5$

$j = \quad 2 \quad 3 \quad {\color{blue}4} \quad 5$

$i = 4 - 2 = 2$

$r[i, j-1] = r[2,3] = 3$

$r[i+1, j] = r[3,4] = 4$

$$W = \begin{bmatrix} 3 & 5 & 13 & 16 & 20 & 22 \\ & 1 & 9 & 12 & 16 & 18 \\ & & 3 & 6 & 10 & 12 \\ & & & 1 & 5 & 7 \\ & & & & 2 & 4 \\ & & & & & 1 \end{bmatrix}$$

$$k = 3 : c[i, k-1] + c[k, j] =$$
$$= c[2,2] + c[3,4] = 5$$

$$C = \begin{bmatrix} & 5 & 18 & & & \\ & & 9 & 18 & & \\ & & & 6 & {\color{red}15} & \\ & & & & 5 & \\ & & & & & 4 \\ & & & & & \end{bmatrix}$$

$$k = 4 : c[i, k-1] + c[k, j] =$$
$$= c[2,3] + c[4,4] = 6$$

$c[i,j] = c[2,4] \leftarrow w[2,4] + 5 =$
15

$r[i,j] = r[2,4] \leftarrow 3$

$$R = \begin{bmatrix} & 1 & 2 & & & \\ & & 2 & 2 & & \\ & & & 3 & {\color{red}3} & \\ & & & & 4 & \\ & & & & & 5 \end{bmatrix}$$

$d = \quad 2 \quad 3 \quad 4 \quad 5$

$j = \quad 2 \quad 3 \quad 4 \quad 5$

$i = 5 - 2 = 3$

$r[i, j-1] = r[3,4] = 4$

$r[i+1, j] = r[4,5] = 5$

$$W = \begin{bmatrix} 3 & 5 & 13 & 16 & 20 & 22 \\ & 1 & 9 & 12 & 16 & 18 \\ & & 3 & 6 & 10 & 12 \\ & & & 1 & 5 & 7 \\ & & & & 2 & 4 \\ & & & & & 1 \end{bmatrix}$$

$k = 4 : c[i, k-1] + c[k, j] =$

$= c[3,3] + c[4,5] = 4$

$$C = \begin{bmatrix} & 5 & 18 & & & \\ & & 9 & 18 & & \\ & & & 6 & 15 & \\ & & & & 5 & 11 \\ & & & & & 4 \\ & & & & & \end{bmatrix}$$

$k = 5 : c[i, k-1] + c[k, j] =$

$= c[3,4] + c[5,5] = 5$

$c[i, j] = c[3,5] \leftarrow w[3,5] + 4 = 11$

$r[i, j] = r[3,5] \leftarrow 4$

$$R = \begin{bmatrix} 1 & 2 & & & \\ & 2 & 2 & & \\ & & 3 & 3 & \\ & & & 4 & 4 \\ & & & & 5 \end{bmatrix}$$

$d = \quad 2 \quad {\color{blue}3} \quad 4 \quad 5$

$j = \quad {\color{blue}3} \quad 4 \quad 5$

$i = 3 - 3 = 0$

$r[i, j-1] = r[0,2] = 2$

$r[i+1, j] = r[1,3] = 2$

$$W = \begin{bmatrix} 3 & 5 & 13 & 16 & 20 & 22 \\ & 1 & 9 & 12 & 16 & 18 \\ & & 3 & 6 & 10 & 12 \\ & & & 1 & 5 & 7 \\ & & & & 2 & 4 \\ & & & & & 1 \end{bmatrix}$$

$$k = 2 : c[i, k-1] + c[k, j] =$$

$$= c[0,1] + c[2,3] = 11$$

$$C = \begin{bmatrix} & 5 & 18 & {\color{red}27} & & \\ & & 9 & 18 & & \\ & & & 6 & 15 & \\ & & & & 5 & 11 \\ & & & & & 4 \\ & & & & & \end{bmatrix}$$

$c[i, j] = c[0,3] \leftarrow$

$w[0,3] + 11 = 27$

$r[i, j] = r[0,3] \leftarrow 2$

$$R = \begin{bmatrix} & 1 & 2 & {\color{red}2} & & \\ & & 2 & 2 & & \\ & & & 3 & 3 & \\ & & & & 4 & 4 \\ & & & & & 5 \end{bmatrix}$$

$d = \quad 2 \quad 3 \quad 4 \quad 5$

$j = \quad 3 \quad 4 \quad 5$

$i = 4 - 3 = 1$

$r[i, j-1] = r[1,3] = 2$

$r[i+1, j] = r[2,4] = 3$

$$W = \begin{bmatrix} 3 & 5 & 13 & 16 & 20 & 22 \\ & 1 & 9 & 12 & 16 & 18 \\ & & 3 & 6 & 10 & 12 \\ & & & 1 & 5 & 7 \\ & & & & 2 & 4 \\ & & & & & 1 \end{bmatrix}$$

$$k = 2 : c[i, k-1] + c[k, j] =$$
$$= c[1,1] + c[2,4] = 15$$

$$C = \begin{bmatrix} 5 & 18 & 27 & & \\ & 9 & 18 & 30 & \\ & & 6 & 15 & \\ & & & 5 & 11 \\ & & & & 4 \end{bmatrix}$$

$$k = 3 : c[i, k-1] + c[k, j] =$$
$$= c[1,2] + c[3,4] = 14$$

$c[i,j] = c[1,4] \leftarrow$

$w[1,4] + 14 = 30$

$r[i,j] = r[1,4] \leftarrow 3$

$$R = \begin{bmatrix} 1 & 2 & 2 & & \\ & 2 & 2 & 3 & \\ & & 3 & 3 & \\ & & & 4 & 4 \\ & & & & 5 \end{bmatrix}$$

$d = \quad 2 \quad {\color{blue}3} \quad 4 \quad 5$

$j = \quad 3 \quad 4 \quad {\color{blue}5}$

$i = 5 - 3 = 2$

$r[i, j-1] = r[2,4] = 3$

$r[i+1, j] = r[3,5] = 4$

$k = 3 : c[i, k-1] + c[k, j] =$

$= c[2,2] + c[3,5] = 11$

$k = 4 : c[i, k-1] + c[k, j] =$

$= c[2,3] + c[4,5] = 10$

$c[i, j] = c[2,5] \leftarrow$
$w[2,5] + 10 = 22$
$r[i, j] = r[2,5] \leftarrow 4$

$$W = \begin{bmatrix} 3 & 5 & 13 & 16 & 20 & 22 \\ & 1 & 9 & 12 & 16 & 18 \\ & & 3 & 6 & 10 & 12 \\ & & & 1 & 5 & 7 \\ & & & & 2 & 4 \\ & & & & & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} & 5 & 18 & 27 & & \\ & & 9 & 18 & 30 & \\ & & & 6 & 15 & {\color{red}22} \\ & & & & 5 & 11 \\ & & & & & 4 \\ & & & & & \end{bmatrix}$$

$$R = \begin{bmatrix} & 1 & 2 & 2 & & \\ & & 2 & 2 & 3 & \\ & & & 3 & 3 & {\color{red}4} \\ & & & & 4 & 4 \\ & & & & & 5 \end{bmatrix}$$

$d = \quad 2 \quad 3 \quad {\color{blue}4} \quad 5$
$j = \quad {\color{blue}4} \quad 5$
$i = 4 - 4 = 0$
$r[i, j-1] = r[0,3] = 2$
$r[i+1, j] = r[1,4] = 3$

$$W = \begin{bmatrix} 3 & 5 & 13 & 16 & 20 & 22 \\ & 1 & 9 & 12 & 16 & 18 \\ & & 3 & 6 & 10 & 12 \\ & & & 1 & 5 & 7 \\ & & & & 2 & 4 \\ & & & & & 1 \end{bmatrix}$$

$k = 2 : c[i, k-1] + c[k, j] =$
$= c[0,1] + c[2,4] = 20$

$k = 3 : c[i, k-1] + c[k, j] =$
$= c[0,2] + c[3,4] = 23$

$$C = \begin{bmatrix} & 5 & 18 & 27 & {\color{red}40} & \\ & & 9 & 18 & 30 & \\ & & & 6 & 15 & 22 \\ & & & & 5 & 11 \\ & & & & & 4 \\ & & & & & \end{bmatrix}$$

$c[i, j] = c[0,4] \leftarrow$
$w[0,4] + 20 = 40$
$r[i, j] = r[0,4] \leftarrow 2$

$$R = \begin{bmatrix} & 1 & 2 & 2 & {\color{red}2} & \\ & & 2 & 2 & 3 & \\ & & & 3 & 3 & 4 \\ & & & & 4 & 4 \\ & & & & & 5 \end{bmatrix}$$

$d = \quad 2 \quad 3 \quad 4 \quad 5$

$j = \quad 4 \quad 5$

$i = 5 - 4 = 1$

$r[i, j-1] = r[1,4] = 3$

$r[i+1, j] = r[2,5] = 4$

$$W = \begin{bmatrix} 3 & 5 & 13 & 16 & 20 & 22 \\ & 1 & 9 & 12 & 16 & 18 \\ & & 3 & 6 & 10 & 12 \\ & & & 1 & 5 & 7 \\ & & & & 2 & 4 \\ & & & & & 1 \end{bmatrix}$$

$k = 3 : c[i, k-1] + c[k, j] =$

$= c[1,2] + c[3,5] = 20$

$$C = \begin{bmatrix} 5 & 18 & 27 & 40 & \\ & 9 & 18 & 30 & 38 \\ & & 6 & 15 & 22 \\ & & & 5 & 11 \\ & & & & 4 \\ & & & & \end{bmatrix}$$

$k = 4 : c[i, k-1] + c[k, j] =$

$= c[1,3] + c[4,5] = 22$

$c[i,j] = c[1,5] \leftarrow$

$w[1,5] + 20 = 38$

$r[i,j] = r[1,5] \leftarrow 3$

$$R = \begin{bmatrix} 1 & 2 & 2 & 2 & \\ & 2 & 2 & 3 & 3 \\ & & 3 & 3 & 4 \\ & & & 4 & 4 \\ & & & & 5 \end{bmatrix}$$

$$d = \quad 2 \quad 3 \quad 4 \quad 5$$
$$j = \quad 5$$
$$i = 5 - 5 = 0$$
$$r[i, j-1] = r[0,4] = 2$$
$$r[i+1, j] = r[1,5] = 3$$

$$W = \begin{bmatrix} 3 & 5 & 13 & 16 & 20 & 22 \\ & 1 & 9 & 12 & 16 & 18 \\ & & 3 & 6 & 10 & 12 \\ & & & 1 & 5 & 7 \\ & & & & 2 & 4 \\ & & & & & 1 \end{bmatrix}$$

$$k = 2 : c[i, k-1] + c[k, j] =$$
$$= c[0,1] + c[2,5] = 27$$

$$C = \begin{bmatrix} & 5 & 18 & 27 & 40 & 49 \\ & & 9 & 18 & 30 & 38 \\ & & & 6 & 15 & 22 \\ & & & & 5 & 11 \\ & & & & & 4 \\ & & & & & \end{bmatrix}$$

$$k = 3 : c[i, k-1] + c[k, j] =$$
$$= c[0,2] + c[3,5] = 29$$

$$c[i,j] = c[0,5] \leftarrow$$
$$w[0,5] + 27 = 49$$
$$r[i,j] = r[0,5] \leftarrow 2$$

$$R = \begin{bmatrix} & 1 & 2 & 2 & 2 & 2 \\ & & 2 & 2 & 3 & 3 \\ & & & 3 & 3 & 4 \\ & & & & 4 & 4 \\ & & & & & 5 \end{bmatrix}$$